



TITLE:

A survey on Stark's conjectures and a result of Dasgupta-Darmon-Pollack (Algebraic Number Theory and Related Topics 2012)

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CITATION:

Kashio, Tomokazu. A survey on Stark's conjectures and a result of Dasgupta-Darmon-Pollack (Algebraic Number Theory and Related Topics 2012). 数理解析研究所講究録別冊 2014, B51: 229-268

ISSUE DATE:

2014-10

URL:

<http://hdl.handle.net/2433/232902>

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A survey on Stark’s conjectures and a result of Dasgupta-Darmon-Pollack

By

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Abstract

This is a survey on Stark’s conjectures and some related topics. We present formulations of Stark’s conjectures, Rubin’s integral refinement for the abelian Stark conjecture, the Brumer-Stark conjecture, and a p -adic analogue of the rank 1 abelian Stark conjecture, which is called the Gross-Stark conjecture. In addition, we describe the recent result [DDP] by Dasgupta-Darmon-Pollack concerning the Gross-Stark conjecture.

Introduction.

In this paper we will survey some results on Stark’s conjecture and its p -adic analogue. Since Stark’s conjecture is a generalization of the class number formula, we start by recalling this formula. Let k be a number field. The Dedekind zeta function is defined by

$$\zeta_k(s) := \prod_{\mathfrak{p} \subset \mathcal{O}_k} (1 - N\mathfrak{p}^{-s})^{-1} \quad (\operatorname{Re}(s) > 1).$$

Here \mathfrak{p} run over all prime ideals of k . We see that it can be extended meromorphically to the whole complex plane \mathbb{C} and is holomorphic at $s = 0$. Then the class number formula states that

$$\zeta_k(s) = \frac{-h_k R_k}{e_k} s^{r_k-1} + O(s^{r_k}) \quad (s \rightarrow 0).$$

Here r_k, h_k, R_k, e_k are the number of infinite places of k , the class number, the regulator, and the number of roots of unity in k , respectively. In particular, we can write

$$\frac{\text{the leading coefficient of } \zeta_k(s) \text{ in the Taylor expansion at } s = 0}{\text{the regulator}} \in \mathbb{Q}.$$

Received March 30, 2013. Revised July 24, 2013, October 3, 2013, November 29, 2013, January 22, 2014 and February 6, 2014.

2010 Mathematics Subject Classification(s): 11M35, 11R23, 11R27, 11R34, 11R42

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Very roughly speaking, the Stark conjecture is its generalization, from Dedekind zeta functions to Artin L functions. In §1, we provide precise statements and some results of Stark's conjectures, the Brumer-Stark conjecture, and Rubin's integral refinement for the rank 1 abelian Stark conjecture. We also deal with a p -adic analogue of the Stark conjecture which is called the Gross-Stark conjecture in §2, §3. In particular, we present recent results [DDP] by Dasgupta-Darmon-Pollack. They proved the Gross-Stark conjecture assuming that Leopoldt's conjecture holds true and that some technical conditions are satisfied. By "cohomological interpretation", we can reduce the Gross-Stark conjecture to the construction of a suitable cocycle. To construct such a cocycle, some techniques of Ribet [Ri] and Wiles [Wi] are used.

Before stating Stark's conjectures, we recall the definition of the regulator R_k for the sake of comparison to the "Stark regulator". Let \mathcal{O}_k be the ring of integers of k , μ_k the group of roots of unity in k , and $\{\infty_1, \dots, \infty_{r_k}\}$ the set of all infinite places of k . Consider the logarithmic embedding of units

$$\lambda: \mathcal{O}_k^\times \rightarrow \mathbb{R}^{r_k}, \quad \lambda(x) := (\log |x|_{\infty_j})_{1 \leq j \leq r_k}.$$

Then Dirichlet's unit theorem states that its image is a free \mathbb{Z} -module of rank $r_k - 1$, and so is $\mathcal{O}_k^\times / \mu_k$. Taking generators $\varepsilon_i \bmod \mu_k \in \mathcal{O}_k^\times / \mu_k$ ($1 \leq i \leq r_k - 1$), we define the regulator R_k of k by

$$R_k := |\det(\log |\varepsilon_i|_{\infty_j})_{1 \leq i, j \leq r_k-1}|$$

$$= \pm \begin{vmatrix} \log |\varepsilon_1|_{\infty_1} & \log |\varepsilon_1|_{\infty_2} & \dots & \log |\varepsilon_1|_{\infty_{r_k-1}} \\ \log |\varepsilon_2|_{\infty_1} & \log |\varepsilon_2|_{\infty_2} & \dots & \log |\varepsilon_2|_{\infty_{r_k-1}} \\ \vdots & \vdots & \ddots & \vdots \\ \log |\varepsilon_{r_k-1}|_{\infty_1} & \log |\varepsilon_{r_k-1}|_{\infty_2} & \dots & \log |\varepsilon_{r_k-1}|_{\infty_{r_k-1}} \end{vmatrix}.$$

Note that the definition of R_k does not depend on the choice of ε_i or the numbering of ∞_j .

§ 1. Stark's conjectures.

Unless otherwise noted, we use the following notations in this paper.

- K/k is a finite Galois extension of number fields with $G := \text{Gal}(K/k)$.
- For a place p of \mathbb{Q} , we denote by S_p the set of all places of k lying above p . In particular, S_∞ is the set of all infinite places. For any set T of places of k , we put

$$T_K := \{w \text{ of } K \mid \exists v \in T \text{ such that } w|v\}.$$

We fix a finite set S of places of k satisfying

$$S_\infty \subset S.$$

- For a \mathbb{Z} -module M and an extension $\mathbb{Z} \subset R$ of rings, we put

$$RM := R \otimes_{\mathbb{Z}} M.$$

§ 1.1. The non-abelian Stark conjecture.

For any \mathbb{C} -valued character χ of G , we will define the following symbols after introducing an isomorphism f of $\mathbb{Q}[G]$ -modules.

- $C_S(\chi) \in \mathbb{C}^\times$: the leading coefficient of the Artin L -function $L_S(s, \chi)$ in the Taylor expansion at $s = 0$ (§1.1.1).
- $R_S(\chi, f) \in \mathbb{C}^\times$: the Stark regulator associated to the group of S -units of K (§1.1.2).
- $A_S(\chi, f) := R_S(\chi, f)/C_S(\chi) \in \mathbb{C}^\times$.

Then Stark's conjecture in the general case is formulated as follows.

Conjecture 1.1. *For any \mathbb{C} -valued character χ of G , we have*

$$(1.1) \quad A_S(\chi, f)^\gamma = A_S(\chi^\gamma, f) \quad (\forall \gamma \in \text{Aut}(\mathbb{C})).$$

Here we put $\chi^\gamma := \gamma \circ \chi$.

Remark. Conjecture (1.1) implies

$$\frac{\text{the leading coefficient of the Artin } L\text{-function } L_S(s, \chi) \text{ at } s = 0}{\text{the Stark regulator}} \in \mathbb{Q}(\chi),$$

where $\mathbb{Q}(\chi) := \mathbb{Q}(\chi(\sigma) \mid \sigma \in F)$.

1.1.1. The leading coefficient $C_S(\chi)$ of the Artin L function $L(s, \chi)$ at $s = 0$.

Let V be the representation space of χ . For each place v of k , we choose a place w of K lying above v , and write its decomposition group, inertia group, Frobenius automorphism as G_v, I_v, Frob_v respectively. Then the S -truncated Artin L function $L_S(s, \chi)$ is defined by

$$L_S(s, \chi) := \prod_{\mathfrak{p} \notin S} \det(1 - \text{Frob}_{\mathfrak{p}} N\mathfrak{p}^{-s}|_{V^{I_{\mathfrak{p}}}})^{-1} \quad (\text{Re}(s) > 1).$$

It can be continued meromorphically to the whole complex plane and is holomorphic at $s = 0$. We denote its leading coefficient and its order of 0 at $s = 0$ by $C_S(\chi)$, $r_S(\chi)$ respectively. That is, we can write

$$L_S(s, \chi) = C_S(\chi)s^{r_S(\chi)} + O(s^{r_S(\chi)+1}) \quad (s \rightarrow 0, C_S(\chi) \neq 0).$$

By using the functional equation for Artin L -functions, we can show the following formula, which we will use later: When $\dim_{\mathbb{C}} V = 1$, we have

$$r_S(\chi) = \begin{cases} |S| - 1 & (\chi = 1_G), \\ |\{v \in S \mid \chi(G_v) = \{1\}\}| & (\chi \neq 1_G). \end{cases}$$

1.1.2. The Stark regulator $R_S(\chi, f)$. Put

$$Y := Y_{S,K} := \bigoplus_{w \in S_K} \mathbb{Z}w.$$

Then G acts on Y in the natural way. We denote by $X = X_{S,K}$ the kernel of $\deg: Y \rightarrow \mathbb{Z}$ ($\sum_w n_w w \mapsto \sum_w n_w$), i.e.,

$$X := X_{S,K} := \left\{ \sum_w n_w w \in Y \mid \sum_w n_w = 0 \right\}.$$

Then we can show that

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\mathbb{C}[G]}(V^*, \mathbb{C}X) = r_S(\chi)$$

with V^* the contragradient representation of V . On the other hand, we denote the group of S -units in K by $U := U_{K,S}$. That is, we can write

$$U := U_{K,S} := \{x \in K^\times \mid |x|_w = 1, \forall w \notin S_K\}.$$

Consider the logarithmic embedding

$$\lambda := \lambda_{K,S}: U \rightarrow \mathbb{R}X$$

which is defined by

$$\lambda(u) := \sum_{w \in S_K} \log |u|_w w.$$

Then Dirichlet's unit theorem states that λ induces the $\mathbb{C}[G]$ -isomorphism

$$\operatorname{id}_{\mathbb{C}} \otimes \lambda: \mathbb{C}U \cong \mathbb{C}X.$$

In particular we see that $\mathbb{Q}[G]$ -modules $\mathbb{Q}U, \mathbb{Q}X$ has the same character. Therefore there exists a (non-canonical) $\mathbb{Q}[G]$ -module isomorphism

$$f: \mathbb{Q}X \cong \mathbb{Q}U.$$

It follows from, for example, [Se, Proposition 32, §12.1]. We consider the automorphism

$$\begin{aligned} (\lambda \circ f)_V: \operatorname{Hom}_{\mathbb{C}[G]}(V^*, \mathbb{C}X) &\rightarrow \operatorname{Hom}_{\mathbb{C}[G]}(V^*, \mathbb{C}X), \\ \phi &\mapsto (\operatorname{id}_{\mathbb{C}} \otimes \lambda) \circ (\operatorname{id}_{\mathbb{C}} \otimes f) \circ \phi. \end{aligned}$$

Now we define the Stark regulator $R_S(\chi)$ by

$$R_S(\chi) = R_S(\chi, f) := \det(\lambda \circ f)_V.$$

1.1.3. Some results for the non-abelian Stark conjecture.

- The truth of Conjecture (1.1) does not depend on the choice of S and f . For the proof, see [Da, §3.6 and Proposition 3.7.2].
- If Conjecture (1.1) holds true for $k = \mathbb{Q}$, then it also holds true for any k as well. For the proof, see [Da, Proposition 3.7.3].
- If Conjecture (1.1) holds true for any abelian extension K/k , then it also holds true for any Galois extension K/k as well. For the proof, see [Da, Proposition 3.7.3].
- If $r_S(\chi) = 0$, then $R_S(\chi) = 1$. Therefore Stark's conjecture (1.1) with $r_S(\chi) = 0$ is equivalent to

$$L_S(0, \chi)^\gamma = L_S(0, \chi^\gamma) \quad (\forall \gamma \in \text{Aut}(\mathbb{C})),$$

which follows from a result of Siegel in [Si].

- When $\chi = 1_G$, Stark's conjecture (1.1) follows from the class number formula. For the proof, see [Da, Proposition 3.7.4].
- When $\mathbb{Q}(\chi) = \mathbb{Q}$ then Stark's conjecture (1.1) holds true. For the proof, see §9 in [Da] or Yamamoto's article (Japanese) in [SS2012].

§ 1.2. The rank 1 abelian Stark conjecture.

In this subsection we introduce a refinement of Conjecture (1.1) under the following additional assumption.

1. K/k is abelian.
2. S contains all ramified places in K/k and all infinite places.
3. S contains a distinguished place v which splits completely in K/k . We fix a place w of K lying above v .
4. $|S| \geq 2$.

Let e_K be the number of roots of unity in K , and \widehat{G} the group of irreducible characters of G . We define the S -truncated partial zeta function $\zeta_S(s, \sigma)$ associated to $\sigma \in G$ by

$$\zeta_S(s, \sigma) := \sum_{(\mathfrak{a}, S)=1, (\mathfrak{a}, K/k)=\sigma} N\mathfrak{a}^{-s}.$$

Here \mathfrak{a} runs over all integral ideals prime to any prime ideal in S whose image under the Artin symbol $(*, K/k)$ is equal to σ . Note that we have

$$\zeta_S(s, \sigma) = \frac{1}{|G|} \sum_{\chi \in \widehat{G}} \overline{\chi}(\sigma) L_S(s, \chi),$$

$$L_S(s, \chi) = \sum_{\sigma \in G} \chi(\sigma) \zeta_S(s, \sigma).$$

Therefore we can show that the assumptions 3,4 implies $\zeta_S(0, \sigma) = 0$ ($\forall \sigma \in G$) by using formulas on $r_S(\chi)$ in §1.1.1.

Conjecture 1.2 ($\text{St}(K/k, S, v, w)$). *Under the assumptions 1,2,3,4, there exists an element $\varepsilon = \varepsilon(K/k, S, v, w) \in K^\times$ satisfying*

- *If $|S| > 2$, then ε is a $\{v\}$ -unit.*
- *If $|S| = 2$, put $S =: \{v, v'\}$. Then ε is an S -unit and $|\varepsilon|_{w'}$ stays constant when $w'|v'$.*
- $\log |\varepsilon^\sigma|_w = -e_K \zeta'_S(0, \sigma)$ ($\forall \sigma \in G$).
- $K(\varepsilon^{1/e_K})/k$ is an abelian extension.

Note that such an element ε is unique up to roots of unity, if it exists. We call Conjecture $\text{St}(K/k, S, v, w)$ the rank 1 abelian Stark conjecture, and the element ε a Stark unit.

1.2.1. Some results for the rank 1 abelian Stark conjecture.

1. If $r_S(\chi) > 1$ for any $\chi \in \widehat{G}$, then $\zeta'_S(0, \sigma) = 0$ for any $\sigma \in G$. In this case, Conjecture $\text{St}(K/k, S, v, w)$ is trivial with $\varepsilon = 1$.
2. The truth of Conjecture $\text{St}(K/k, S, v, w)$ does not depend on the choice of v, w (for the proof, see [Da, Remark 4.3.3, Proposition 4.3.4]). So we may write Conjecture $\text{St}(K/k, S, v)$ or Conjecture $\text{St}(K/k, S)$.
3. Under the assumptions 1,2,3,4, Conjecture $\text{St}(K/k, S)$ implies Conjecture (1.1) for all χ with $r_S(\chi) = 1$. For the proof, see §4 in [Da] or the author's article (Japanese) in [SS2012].
4. Conjecture $\text{St}(K/k, S)$ is related to Hilbert's 12th problem. Assume that G is cyclic and v is the only place in S which splits completely in K/k . In this case, Conjecture $\text{St}(K/k, S, v, w)$ implies $K = k(\varepsilon)$. Additionally assume that v is real. We may regard that k, K are subfields of \mathbb{R} , that is, $k \subset K \subset K_w = \mathbb{R}$. Then the Stark unit ε is given by $\varepsilon = \exp(-2\zeta'_S(0, \text{id}))$ [Da, Remark 4.3.3].

5. If $|G| = 2$ then Conjecture $\text{St}(K/k, S)$ holds true. There is a partial result by Sands when G has exponent 2. For detail, see §7 in [Da].
6. When $k = \mathbb{Q}$ or an imaginary quadratic field, Conjecture $\text{St}(K/k, S)$ holds true. For the proof, see [Ta1], [St], or [SS2012, the author's article (Japanese) in the case of $k = \mathbb{Q}$, $v = \infty$, Onodera's article (Japanese) in the case of imaginary quadratic fields]. There is a sketch of a proof in the case of $k = \mathbb{Q}$ below.
7. When k is a real quadratic field, Shintani independently formulated a conjecture which is almost equivalent to Conjecture $\text{St}(K/k, S)$ in [Shin].

For example let $k = \mathbb{Q}$, $K := \mathbb{Q}(\zeta_m + \zeta_m^{-1})$ ($\zeta_m := \exp(\frac{2\pi i}{m})$), $S := \{p|m\} \cup \{\infty\}$, $v := \infty$. Define an element $\sigma_{\pm a} \in G = \text{Gal}(K/\mathbb{Q})$ by

$$\sigma_{\pm a}: \zeta_m + \zeta_m^{-1} \mapsto \zeta_m^a + \zeta_m^{-a}$$

for $a \in \mathbb{Z}$ with $0 < a < m$, $(a, m) = 1$. Then we can write

$$\zeta_S(s, \sigma_{\pm a}) = \zeta(s, m, a) + \zeta(s, m, m - a)$$

by using the Hurwitz zeta function

$$\zeta(s, m, a) := \sum_{n=0}^{\infty} (a + nm)^{-s}.$$

Recall Lerch's formula

$$\zeta'(0, m, a) = \log\left(\Gamma\left(\frac{a}{m}\right)(2\pi)^{-\frac{1}{2}} m^{\frac{2a-m}{2m}}\right)$$

and Euler's formulas

$$\begin{aligned} \frac{\pi}{\sin(z\pi)} &= \Gamma(z)\Gamma(1-z), \\ \sin(z) &= \frac{\exp(zi) - \exp(-zi)}{2i}. \end{aligned}$$

Then we see that

$$\exp(-2\zeta'_S(0, \sigma_{\pm a})) = \zeta_m^a + \zeta_m^{-a} - 2 = -\zeta_m^a(1 - \zeta_m^{-a})^2.$$

Therefore we see that $\text{St}(\mathbb{Q}(\zeta_m + \zeta_m^{-1})/\mathbb{Q}, S, \infty)$ holds true with $\varepsilon = \zeta_m + \zeta_m^{-1} - 2$ and that the Stark unit in this case is essentially equal to a cyclotomic unit. On the other hand, Stark units in the case of $k = \mathbb{Q}$, $v = p < \infty$ are essentially equal to (products of) Gauss sums. For example, let $K/k := \mathbb{Q}(\zeta_{p-1})/\mathbb{Q}$, $S := \{l|p(p-1)\} \cup \{\infty\}$ ($p \neq 2, 3$).

Then $v := p$ splits completely in $\mathbb{Q}(\zeta_{p-1})/\mathbb{Q}$. Fix a prime ideal $w := \mathfrak{P}$ of $\mathbb{Q}(\zeta_{p-1})$ lying above p . Define an element $\sigma_a \in G = \text{Gal}(\mathbb{Q}(\zeta_{p-1})/\mathbb{Q})$ by

$$\sigma_a: \zeta_{p-1} \mapsto \zeta_{p-1}^a$$

for $a \in \mathbb{Z}$ with $0 < a < p-1$, $(a, p-1) = 1$. Since $\zeta_S(s, \sigma_a) = (1 - p^{-s})\zeta_{S-\{p\}}(s, \sigma_a)$, $(1 - p^{-s})|_{s=0} = 0$, $\frac{d}{ds}(1 - p^{-s})|_{s=0} = \log p$, $\zeta_{S-\{p\}}(0, \sigma_a) = \zeta(0, p-1, a) = \frac{1}{2} - \frac{a}{p-1}$, we have

$$\zeta'_S(0, \sigma_a) = \log p \cdot \left(\frac{1}{2} - \frac{a}{p-1} \right).$$

There exists a unique homomorphism $\chi = \chi_{\mathfrak{P}}: (\mathbb{Z}/p\mathbb{Z})^\times \rightarrow \mathbb{Z}[\zeta_{p-1}]^\times$ satisfying

$$\chi(x) \bmod \mathfrak{P} = x \text{ (as an element in } \mathbb{Z}/p\mathbb{Z} = \mathbb{Z}[\zeta_{p-1}]/\mathfrak{P})$$

for all $x \in (\mathbb{Z}/p\mathbb{Z})^\times$. We put

$$G(\chi) := \sum_{x \in (\mathbb{Z}/p\mathbb{Z})^\times} \chi(x) \zeta_p^x.$$

Then we can show that

$$\varepsilon := \frac{G(\chi)^{p-1}}{p^{\frac{p-1}{2}}} \in \mathbb{Q}(\zeta_{p-1})$$

satisfies the condition of $\text{St}(\mathbb{Q}(\zeta_{p-1})/\mathbb{Q}, S, p)$.

§ 1.3. Rubin's integral refinement for the abelian Stark conjecture.

Assume that K/k is abelian and put \widehat{G} to be the group of all irreducible characters of G . For $\chi \in \widehat{G}$, we define

$$e_\chi := \frac{1}{|G|} \sum_{\sigma \in G} \overline{\chi}(\sigma) \sigma \in \mathbb{C}[G].$$

Note that e_χ is the idempotent associated to χ . Then related to the statement (1.1) of Stark's conjecture, we have the following equivalence.

$$\begin{aligned} A_S(\chi, f)^\gamma &= A_S(\chi^\gamma, f) \quad (\forall \gamma \in \text{Aut}(\mathbb{C}), \forall \chi \in \widehat{G}) \\ &\Leftrightarrow \sum_{\chi \in \widehat{G}} A_S(\chi, f) \chi(\sigma) \in \mathbb{Q} \quad (\forall \sigma \in G) \\ &\Leftrightarrow \sum_{\chi \in \widehat{G}} A_S(\chi, f) e_{\overline{\chi}} \in \mathbb{Q}[G]. \end{aligned}$$

Therefore, it is natural to consider the “ G -equivariant L -function”

$$\sum_{\chi \in \widehat{G}} L_S(s, \chi) e_{\overline{\chi}}$$

and study the ratio

$$\frac{\text{the leading coefficient of the } G\text{-equivariant } L\text{-function at } s = 0}{\text{a “}G\text{-equivariant regulator”}}.$$

In this subsection we present Rubin's conjecture on such ratios, which is a refinement of Stark's conjecture (1.1) in the case of abelian extensions K/k , and is a generalization of Conjecture $\text{St}(K/k, S)$ to the higher order case.

1.3.1. Rubin's integral refinement. Let K/k be a finite abelian extension of number fields with $G = \text{Gal}(K/k)$. (We can formulate Rubin's conjecture in the case of global function fields similarly.) We denote by \widehat{G} , μ_K the group of irreducible characters of G , the group of roots of unity in K respectively. Different from previous sections,

we take two finite and non-empty sets S, T of places of k .

For $r = 0, 1, 2, \dots$, we consider the following assumption.

Definition 1.3 (Assumption (H_r)).

1. S contains all infinite places of k and all ramified places in K/k .
2. $T \cap S = \emptyset$.
3. $\{\zeta \in \mu_K \mid \zeta \equiv 1 \pmod{T_K}\} = \{1\}$.
4. S contains more than or equal to r places which split completely in K/k .
5. $|S| \geq r + 1$.

Here we define $x \equiv 1 \pmod{T_K}$ by $x \equiv 1 \pmod{w}$ ($\forall w \in T_K$).

Remark. Assumption (H_r) -4,5 implies $r_S(\chi) = \text{ord}_{s=0} L_S(s, \chi) \geq r$ ($\forall \chi \in \widehat{G}$).

Definition 1.4. Let $e_\chi := \frac{1}{|G|} \sum_{\sigma \in G} \overline{\chi}(\sigma) \sigma \in \mathbb{C}[G]$. We put

$$\begin{aligned} \Theta_S(s) &:= \sum_{\chi \in \widehat{G}} L_S(s, \chi) e_{\overline{\chi}}, \\ \Theta_{S,T}(s) &:= \left(\prod_{\mathfrak{p} \in T} (1 - \text{Frob}_{\mathfrak{p}}^{-1} N\mathfrak{p}^{1-s}) \right) \Theta_S(s). \end{aligned}$$

Then $\Theta_S(s)$ (resp. $\Theta_{S,T}(s)$) is a $\mathbb{C}[G]$ -valued meromorphic (resp. holomorphic) function. Under Assumption (H_r) , we put

$$\begin{aligned} \Theta_S^{(r)}(0) &:= \lim_{s \rightarrow 0} \frac{\Theta_S(s)}{s^r}, \\ \Theta_{S,T}^{(r)}(0) &:= \lim_{s \rightarrow 0} \frac{\Theta_{S,T}(s)}{s^r}. \end{aligned}$$

Remark. We can write

$$\Theta_{S,T}^{(r)}(0) = \delta_T \Theta_S^{(r)}(0)$$

with $\delta_T := \prod_{\mathfrak{p} \in T} (1 - \text{Frob}_{\mathfrak{p}}^{-1} N\mathfrak{p}) \in \mathbb{Q}[G]^\times$.

As usual we denote the ring of S -integers of K by

$$\mathcal{O}_S := \mathcal{O}_{K,S} := \{x \in K \mid |x|_w \leq 1 \ (\forall w \notin S_K)\}.$$

We put

$$U_S := U_{K,S} := \mathcal{O}_{K,S}^\times,$$

and

$$U_{S,T} := U_{K,S,T} := \{x \in U_{K,S} \mid x \equiv 1 \pmod{T_K}\}.$$

Note that $[U_S : U_{S,T}] < \infty$.

Remark. Assumption (H_r) -3 implies $U_{S,T}$ is torsion-free.

Definition 1.5. Under Assumption (H_r) , we take r places $v_1, \dots, v_r \in S$ which split completely in K/k . Choose a place w_i of K dividing v_i for each i . We define the G -equivariant regulator map

$$R_W \in \text{Hom}_{\mathbb{Z}[G]} \left(\bigwedge_{\mathbb{Z}[G]}^r U_{S,T}, \mathbb{C}[G] \right)$$

with respect to $W := (w_1, w_2, \dots, w_r)$ by

$$R_W(u_1 \wedge \cdots \wedge u_r) := \det \left(\left(- \sum_{\sigma \in G} \log |u_i^{\sigma^{-1}}|_{w_j \sigma} \right)_{1 \leq i, j \leq r} \right).$$

We put

$$e_{S,r} := \sum_{\chi \in \widehat{G}, \ r_S(\chi)=r} e_\chi.$$

We see that $e_{S,r} \in \mathbb{Q}[G]$ since for all $\gamma \in \text{Aut}(\mathbb{C})$, we have $r_S(\chi) = r_S(\chi^\gamma)$. It is clear that Assumption (H_r) implies

$$\Theta_{S,T}^{(r)}(0) \in e_{S,r} \mathbb{C}[G].$$

Moreover we can show that Dirichlet's unit theorem gives the $\mathbb{C}[G]$ -isomorphism

$$\text{id}_{\mathbb{C}} \otimes R_W : e_{S,r} \left(\mathbb{C} \bigwedge_{\mathbb{Z}[G]}^r U_{S,T} \right) \cong e_{S,r} \mathbb{C}[G].$$

Therefore we can define

$$\varepsilon_{S,T,r} := (\mathrm{id}_{\mathbb{C}} \otimes R_W)^{-1} \left(\Theta_{S,T}^{(r)}(0) \right) \in e_{S,r} \left(\mathbb{C} \bigwedge_{\mathbb{Z}[G]}^r U_{S,T} \right).$$

Remark. We can write the relation between Stark's regulator $R_S(\chi, f)$ and Rubin's G -equivariant regulator map R_W as follows: We can take $f: \mathbb{Q}X_S \cong \mathbb{Q}U_S$ and $\varepsilon_i \in e_{S,r} \mathbb{Q}U_S$ so that

$$(\mathrm{id}_{\mathbb{C}} \otimes R_W)(\varepsilon_1 \wedge \cdots \wedge \varepsilon_r) = \sum_{\chi \in \widehat{G}, r_S(\chi)=r} R_S(\chi, f) e_{\overline{\chi}}.$$

We note that this relation “corresponds” to

$$\begin{aligned} \Theta_S^{(r)}(0) &= \sum_{\chi \in \widehat{G}} \left(\lim_{s \rightarrow 0} \frac{L_S(s, \chi)}{s^r} \right) e_{\overline{\chi}} \\ &= \sum_{\chi \in \widehat{G}, r_S(\chi)=r} \text{“the leading coefficient of } L_S(s, \chi)\text{”} e_{\overline{\chi}}. \end{aligned}$$

Associated to $\Phi := (\phi_1, \dots, \phi_{r-1}) \in \mathrm{Hom}_{\mathbb{Z}[G]}(U_{S,T}, \mathbb{Z}[G])^{r-1}$, we define

$$\begin{aligned} \widetilde{\Phi} &\in \mathrm{Hom}_{\mathbb{Q}[G]} \left(\mathbb{Q} \bigwedge_{\mathbb{Z}[G]}^r U_{S,T}, \mathbb{Q}U_{S,T} \right), \\ \widetilde{\Phi}(u_1 \wedge \cdots \wedge u_r) &:= \sum_{k=1}^r (-1)^k \det \left((\phi_i(u_j))_{j \neq k} \right) u_k. \end{aligned}$$

Here $(\phi_i(u_j))_{j \neq k}$ is a matrix of the size $(r-1) \times (r-1)$ where i runs over the range $1 \leq i \leq r-1$, j runs over the range $1 \leq j \leq r, j \neq k$.

Definition 1.6.

$$\Lambda_{S,T,r} := \left\{ \varepsilon \in e_{S,r} \left(\mathbb{Q} \bigwedge_{\mathbb{Z}[G]}^r U_{S,T} \right) \mid \widetilde{\Phi}(\varepsilon) \in U_{S,T} \ (\forall \Phi \in \mathrm{Hom}_{\mathbb{Z}[G]}(U_{S,T}, \mathbb{Z}[G])^{r-1}) \right\}.$$

We now state Rubin's integral refinement for the abelian Stark conjecture.

Conjecture 1.7. *Under Assumption (H_r) , we have*

$$(1.2) \quad \varepsilon_{S,T,r} \in \Lambda_{S,T,r}.$$

Remark. For any abelian extension K/k of number fields, the following are equivalent.

1. Conjecture (1.1) holds true for any χ with $r_S(\chi) = r$.
2. Under Assumption (H_r) , we have $\varepsilon_{S,T,r} \in e_{S,r} \left(\mathbb{Q} \bigwedge_{\mathbb{Z}[G]}^r U_{S,T} \right) = \mathbb{Q} \otimes_{\mathbb{Z}} \Lambda_{S,T,r}$.

Therefore the above conjecture is called Rubin's integral refinement for the abelian Stark conjecture.

1.3.2. Some results for Rubin's integral refinement.

1. The truth of Rubin's conjecture (1.2) does not depend on the choice of places $W = (w_1, \dots, w_r)$ [BPSS, Remark 2, §2.1 in Popescu's article].
2. When K/k is a quadratic extension of number fields, Rubin's conjecture (1.2) holds true [Ru, Theorem 2.5].
3. Under Assumption (H_r) -1,4,5, Rubin's conjecture (1.2) implies Stark's conjecture (1.1) for all χ with $r_S(\chi) = r$. For the proof, see [Ru] or the author's article in [SS2012] (Japanese).
4. When $r = 0$, Rubin's conjecture (1.2) only states $\Theta_{S,T}(0) \in \mathbb{Z}[G]$. It follows from a result of Deligne-Ribet.
5. When $r = 1$, the following are equivalent [Ru, Proposition 2.5].
 - (a) Rubin's conjecture (1.2) with $r = 1$ holds true for all T .
 - (b) Conjecture $\text{St}(K/k, S)$ holds true.
6. In the case of $r = 1$ and $v_1 < \infty$, Rubin's conjecture (1.2) is equivalent to the Brumer-Stark conjecture and there are some partial results. We will formulate the Brumer-Stark conjecture in the next subsection. There is a survey on this topic by Miura in [SS2012] (Japanese).

§ 1.4. The Brumer-Stark conjecture.

In this subsection, we study Conjecture $\text{St}(K/k, S, v, w)$ in the case of finite places v, w . We write $v = \mathfrak{p}$, $w = \mathfrak{P}$. We assume that $|S| > 2$ and put $R := S - \{\mathfrak{p}\}$. Since \mathfrak{p} splits completely, we see $(\mathfrak{p}, K/k) = \text{id}_K$. Therefore we have

$$\begin{aligned}
 \zeta_S(s, \sigma) &= \sum_{(\mathfrak{a}, S)=1, (\mathfrak{a}, K/k)=\sigma} N\mathfrak{a}^{-s} \\
 &= \sum_{(\mathfrak{a}, R)=1, (\mathfrak{a}, K/k)=\sigma} N\mathfrak{a}^{-s} - \sum_{(\mathfrak{a}, R)=1, \mathfrak{p}|\mathfrak{a}, (\mathfrak{a}, K/k)=\sigma} N\mathfrak{a}^{-s} \\
 &= (1 - N\mathfrak{p}^{-s})\zeta_R(s, \sigma).
 \end{aligned}$$

So we get

$$\zeta'_S(0, \sigma) = \zeta_R(0, \sigma) \cdot \log N\mathfrak{p}.$$

Hence Conjecture $\text{St}(K/k, S, \mathfrak{p}, \mathfrak{P})$ implies that

$$\theta\mathfrak{P} \text{ is a principal ideal } (= (\varepsilon))$$

for

$$\theta := \sum_{\sigma \in G} e_K \zeta_R(0, \sigma) \sigma^{-1}.$$

We note that we have $e_K \zeta_R(0, \sigma) \in \mathbb{Z}$ with $e_K := |\mu_K|$ by a result of Deligne-Ribet and that the action of $\sum_{\sigma \in G} n_\sigma \sigma \in \mathbb{Z}[G]$ is defined by $(\sum_{\sigma \in G} n_\sigma \sigma) \mathfrak{a} := \prod_{\sigma \in G} (\sigma(\mathfrak{a}))^{n_\sigma}$. Moreover, we can show that the truth of Conjecture $\text{St}(K/k, S, \mathfrak{p}, \mathfrak{P})$ for all $\mathfrak{p}, \mathfrak{P}$ is equivalent to the truth of the following conjecture, which is called the Brumer-Stark conjecture.

Conjecture 1.8. *Let the notation be as in §1.3.1. Additionally let $e_K := |\mu_K|$, π the natural map $K^\times \rightarrow \mathbb{Q}K^\times$, and*

$$K_{S,0}^\times := \{x \in K^\times \mid \pi(x) \in e_{S,0} \mathbb{Q}K^\times\}.$$

Then under Assumption (H_0) -1, for any fractional ideal $I \subset K$, there exists $\alpha_I \in K_{S,0}^\times$ satisfying

$$\begin{aligned} e_K \Theta_S(0) I &= (\alpha_I), \\ K(\alpha_I^{1/e_K})/k &\text{ is abelian.} \end{aligned}$$

Remark. There exists a Strong version of the Brumer-Stark conjecture. Let $A_{K,S,T}$ be the (S, T) -modified ideal class group. That is

$$A_{K,S,T} := \frac{\{\text{fractional ideals } \mathfrak{a} \text{ of } \mathcal{O}_{K,S} \mid (\mathfrak{a}, T_K) = 1\}}{\{\text{principal ideals } (x) \text{ of } \mathcal{O}_{K,S} \mid x \equiv 1 \pmod{T_K}\}}.$$

Then the Strong Brumer-Stark “conjecture” states that

$$\text{Under the assumption } (H_0), \text{ we have } \Theta_{S,T}(0) \in (\mathbb{Z}[G] \cap e_{S,0} \mathbb{Q}[G]) \cdot \text{Fitt}_{\mathbb{Z}[G]}(A_{K,S,T}).$$

Here we denote the Fitting ideal by Fitt . Noting that

$$\text{Fitt}_{\mathbb{Z}[G]}(A_{S,T}) \subset \text{Ann}_{\mathbb{Z}[G]}(A_{S,T}),$$

we can show that the Strong Brumer-Stark “conjecture” implies the Brumer-Stark conjecture. There are some partial results and some counterexamples for the Strong version.

§ 2. The Gross-Stark conjecture.

In this section, we present a formulation of the Gross-Stark conjecture, which is a p -adic analogue of the rank 1 abelian Stark conjecture. Hereafter we denote H/F instead of K/k as in [DDP]. Without loss of generality, we may assume that

- F is a totally real field of degree n , H is a totally complex field, and H/F is a cyclic extension of conductor \mathfrak{n} .
- The character $\chi: G := \text{Gal}(H/F) \rightarrow \overline{\mathbb{Q}}^\times$ is injective.

Take a rational prime p and fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p} \hookrightarrow \mathbb{C}$ for simplicity. Let $E := \mathbb{Q}_p(\chi(\sigma) \mid \sigma \in G)$ and ω the Teichmüller character. We denote the set of primes of F dividing p by S_p and assume that

$$S_p \subset S.$$

Then the p -adic L -function $L_{S,p}(s, \chi\omega)$ is characterized by the following.

There exists a unique E -valued continuous function $L_{S,p}(s, \chi\omega)$ satisfying

$$L_{S,p}(n, \chi\omega) = L_S(n, \chi\omega^n) \quad (\forall n \leq 0).$$

We can also show that $L_{S,p}(s, \chi\omega)$ is holomorphic at $s = 0$. Moreover Gross conjectured ([Gr, Conjecture 2.12])

$$\text{ord}_{s=0} L_{S,p}(s, \chi\omega) = r_S(\chi).$$

We consider the simplest case. Namely assume that

$$S := S_\infty \cup S_p \cup \{v \mid \mathfrak{n}\}.$$

Then the above conjecture states

$$\text{ord}_{s=0} L_{S,p}(s, \chi\omega) = |\{\mathfrak{q} \in S_p \mid \chi(\mathfrak{q}) = 1\}|.$$

Remark. Put $R := S - \{\mathfrak{q} \in S_p \mid \chi(\mathfrak{q}) = 1\}$. Then we can write

$$L_S(s, \chi) = \left(\prod_{\mathfrak{q} \in S_p, \chi(\mathfrak{q})=1} (1 - N\mathfrak{q}^{-s}) \right) L_R(s, \chi).$$

Therefore, $L_S(s, \chi)$ has a “trivial zero” of order $r_S(\chi) = |\{\mathfrak{q} \in S_p \mid \chi(\mathfrak{q}) = 1\}|$ at $s = 0$. However, it is non-trivial whether $L_{S,p}(s, \chi\omega)$ has a zero of the same order.

The following condition is essential for the Gross-Stark conjecture: We assume that $r_S(\chi) = 1$, i.e.,

there exists a unique prime ideal $\mathfrak{p} \in S_p$ satisfying $\chi(\mathfrak{p}) = 1$.

In particular, we see that \mathfrak{p} splits completely in H/F and that $\text{ord}_{s=0} L_{S,p}(s, \chi\omega) \geq 1$. In this case, Dirichlet's unit theorem states that we have

$$\dim_E U_\chi = r_S(\chi) = 1$$

with

$$U_\chi := (\mathcal{O}_{H,S}^\times \otimes_{\mathbb{Z}} E)^{\chi^{-1}} = \{u \in \mathcal{O}_{H,S}^\times \otimes_{\mathbb{Z}} E \mid \sigma u = \chi^{-1}(\sigma)u \ (\forall \sigma \in G)\}.$$

Take a non-zero element u_χ of U_χ and a prime ideal \mathfrak{P} of H dividing \mathfrak{p} . We define E -linear maps $\text{ord}_{\mathfrak{P}}, \mathbf{L}_{\mathfrak{P}}$ by

$$\begin{aligned} \text{ord}_{\mathfrak{P}}: U_\chi &\rightarrow E, \ \varepsilon \otimes \alpha \mapsto \alpha \cdot \text{ord}_{\mathfrak{P}} \varepsilon, \\ \mathbf{L}_{\mathfrak{P}}: U_\chi &\rightarrow E, \ \varepsilon \otimes \alpha \mapsto \alpha \cdot \log_p(N_{H_{\mathfrak{P}}/\mathbb{Q}_p} \varepsilon). \end{aligned}$$

Then the \mathcal{L} -invariant $\mathcal{L}(\chi)$ is defined by

$$\mathcal{L}(\chi) := -\frac{\mathbf{L}_{\mathfrak{P}}(u_\chi)}{\text{ord}_{\mathfrak{P}}(u_\chi)}.$$

Remark. We can show that the value $\mathcal{L}(\chi) \in E$ does not depend on the choice of u_χ, \mathfrak{P} . For example, write $\mathfrak{P}^{h_K} = (\pi)$ with $\pi \in H$, h_K the class number of K . Then we see that $U_\chi \ni \sum_{\sigma \in G} \chi(\sigma) \otimes \pi^\sigma \neq 0$. Therefore there exists $t \in E^\times$ satisfying

$$u_\chi = t \sum_{\sigma \in G} \chi(\sigma) \otimes \pi^\sigma.$$

It is clear that

$$\begin{aligned} \mathbf{L}_{\mathfrak{P}}(u_\chi) &= t \sum_{\sigma \in G} \chi(\sigma) \log_p(N_{H_{\mathfrak{P}}/\mathbb{Q}_p} \pi^\sigma), \\ \text{ord}_{\mathfrak{P}}(u_\chi) &= th_K. \end{aligned}$$

Hence we get

$$\mathcal{L}(\chi) = \frac{\sum_{\sigma \in G} \chi(\sigma) \log_p(N_{H_{\mathfrak{P}}/\mathbb{Q}_p} \pi^\sigma)}{h_K},$$

which does not depend on the choice of u_χ . By a similar argument, we can prove that it does not depend on the choice of \mathfrak{P} , either.

We now state the Gross-Stark conjecture.

Conjecture 2.1 ([Gr], Conjecture 3.13). *Let F be a totally real field, H a totally complex field with H/F a cyclic extension of conductor \mathfrak{n} . Assume that the character $\chi: G = \text{Gal}(H/F) \rightarrow \overline{\mathbb{Q}}^\times$ is injective, $S = S_\infty \cup S_p \cup \{v|\mathfrak{n}\}$, and $r_S(\chi) = 1$. Then we have*

$$L'_{S,p}(0, \chi\omega) = \mathcal{L}(\chi) L_R(0, \chi).$$

§ 3. A result of Dasgupta-Darmon-Pollack.

To describe the main result of [DDP], we need the following notations.

Definition 3.1.

$$\begin{aligned}\mathcal{L}_{\text{an}}(s, \chi) &:= \frac{-L_{S,p}(1-s, \chi\omega)}{L_R(0, \chi)}, \\ \mathcal{L}_{\text{an}}(\chi) &:= \frac{L'_{S,p}(0, \chi\omega)}{L_R(0, \chi)} = \mathcal{L}'_{\text{an}}(1, \chi).\end{aligned}$$

Now here is the main Theorem.

Theorem 3.2 ([DDP], Theorem 2). *Assume the following.*

- (3.1) *If $|S_p| > 1$, assume that Leopoldt's conjecture is true for F .
If $|S_p| = 1$, assume that Leopoldt's conjecture is true for F ,
and that $\text{ord}_{s=1}(\mathcal{L}_{\text{an}}(s, \chi) + \mathcal{L}_{\text{an}}(s, \chi^{-1})) = \text{ord}_{s=1}\mathcal{L}_{\text{an}}(s, \chi^{-1})$.*

Then the Gross-Stark conjecture holds true.

§ 3.1. Cohomological interpretation.

We reformulate the Gross-Stark conjecture in terms of Galois cohomology in this subsection. To do this, we will use the following notations.

- We put $G_H := \text{Gal}(\overline{F}/H) \subset G_F := \text{Gal}(\overline{F}/F)$, $G_{F_v} := \text{Gal}(\overline{F}_v/F_v)$ (for each place v of F). Then we may regard $G = \text{Gal}(H/F) = G_F/G_H$, $G_{F_v} \subset G_F$.
- We put $\mu_n := \{\zeta \in \overline{F} \mid \zeta^n = 1\}$, $\epsilon_{\text{cyc}}: G_F \rightarrow \mathbb{Z}_p^\times$ to be the cyclotomic character. That is, we have

$$\sigma(\zeta) = \zeta^{\epsilon_{\text{cyc}}(\sigma)}$$

for all $\zeta \in \mu_{p^n}$, $n \in \mathbb{N}$.

- We denote by $E(\chi^{-1})$, $E(1)$, $E(1)(\chi)$ the representation spaces over E of characters χ^{-1} , ϵ_{cyc} , $\chi\epsilon_{\text{cyc}}$ respectively. As E -vector spaces, $E(\chi^{-1}) = E(1) = E(1)(\chi) = E$. For any E -vector space V , we put $V(\chi^{-1}) := V \otimes E(\chi^{-1})$, etc.
- We define elements $\kappa_{\text{nr}} \in \text{Hom}(G_{F_v}, E)$, $\kappa_{\text{cyc}} \in \text{Hom}(G_F, E)$ as follows.

$$\kappa_{\text{nr}} \text{ is unramified and } \kappa_{\text{nr}}(\text{Frob}_v) := 1,$$

$$\kappa_{\text{cyc}} := \log_p \circ \epsilon_{\text{cyc}}.$$

When $\chi|_{G_{F_v}} = 1$, we use the same symbols κ_{nr} , κ_{cyc} for the corresponding elements in $H^1(G_{F_v}, E(\chi^{-1})) = \text{Hom}_{\text{conti}}(G_{F_v}, E)$.

Moreover, we will use the following well-known results. For each place v of F , we consider the perfect pairing

$$\langle \cdot, \cdot \rangle_v: H^1(G_{F_v}, E(\chi^{-1})) \times H^1(G_{F_v}, E(1)(\chi)) \rightarrow E,$$

which is defined by “Tate’s local duality” in §3.1.1. Then the global reciprocity law of class field theory gives the following relation.

$$\forall \kappa \in H^1(G_F, E(\chi^{-1})), \forall u \in H^1(G_F, E(1)(\chi)), \langle \kappa, u \rangle := \sum_v \langle \text{res}_v \kappa, \text{res}_v u \rangle_v = 0.$$

Here we put $\text{res}_v f := f|_{G_{F_v}}$. As we will see in §3.1.2, we can embed

$$\delta: U_\chi \hookrightarrow H^1(G_F, E(1)(\chi))$$

by using “Kummer theory”. In [DDP], the subspace

$$H_{\mathfrak{p}, \text{cyc}}^1(G_F, E(\chi^{-1})) \subset H^1(G_F, E(\chi^{-1})),$$

which is characterized by (3.4), is constructed. This subspace “corresponds” to

$$\delta(U_\chi) \subset H^1(G_F, E(1)(\chi))$$

in the following sense:

- $\forall v \neq \mathfrak{p}, \text{res}_v(H_{\mathfrak{p}, \text{cyc}}^1(G_F, E(\chi^{-1}))) \perp \text{res}_v(\delta(U_\chi))$ w.r.t. $\langle \cdot, \cdot \rangle_v$. That is,

$$\langle \text{res}_v \kappa, \text{res}_v u \rangle_v = 0 \quad (\forall v \neq \mathfrak{p}, \forall \kappa \in H_{\mathfrak{p}, \text{cyc}}^1(G_F, E(\chi^{-1})), \forall u \in \delta(U_\chi)).$$

Moreover, it satisfies the following properties:

- $\text{res}_{\mathfrak{p}}: H_{\mathfrak{p}, \text{cyc}}^1(G_F, E(\chi^{-1})) \hookrightarrow E \cdot \kappa_{\text{nr}} \oplus E \cdot \kappa_{\text{cyc}}$. That is,

$$\begin{aligned} & \text{res}_{\mathfrak{p}} \text{ on } H_{\mathfrak{p}, \text{cyc}}^1(G_F, E(\chi^{-1})) \text{ is injective and} \\ & \text{res}_{\mathfrak{p}}(H_{\mathfrak{p}, \text{cyc}}^1(G_F, E(\chi^{-1}))) \subset E \cdot \kappa_{\text{nr}} \oplus E \cdot \kappa_{\text{cyc}}. \end{aligned}$$

- $\dim_E H_{\mathfrak{p}, \text{cyc}}^1(G_F, E(\chi^{-1})) = 1$.

Therefore the Gross-Stark conjecture is equivalent to the following conjecture.

Conjecture 3.3. *There exists a non-trivial element $\kappa \in H_{\mathfrak{p}, \text{cyc}}^1(G_F, E(\chi^{-1}))$ satisfying the following.*

$$\text{Write } \text{res}_{\mathfrak{p}} \kappa = x \cdot \kappa_{\text{nr}} + y \cdot \kappa_{\text{cyc}} \text{ with } x, y \in E. \text{ Then } \mathcal{L}_{\text{an}}(\chi) = -x/y.$$

We can see the equivalence of two conjectures as follows: Put $\text{res}_{\mathfrak{p}}\kappa = x \cdot \kappa_{\text{nr}} + y \cdot \kappa_{\text{cyc}}$ for $0 \neq \kappa \in H_{\mathfrak{p}, \text{cyc}}^1(G_F, E(\chi^{-1}))$. Then we have

$$\begin{aligned} 0 = \langle \delta(u_\chi), \kappa \rangle &= \sum_v \langle \text{res}_v(\delta(u_\chi)), \text{res}_v(\kappa) \rangle_v \\ &= \langle \text{res}_{\mathfrak{p}}(\delta(u_\chi)), \text{res}_{\mathfrak{p}}(\kappa) \rangle_{\mathfrak{p}} \\ &= x \langle \text{res}_{\mathfrak{p}}(\delta(u_\chi)), \kappa_{\text{nr}} \rangle_{\mathfrak{p}} + y \langle \text{res}_{\mathfrak{p}}(\delta(u_\chi)), \kappa_{\text{cyc}} \rangle_{\mathfrak{p}}. \end{aligned}$$

On the other hand, the reciprocity law of local class field theory states that

$$\begin{aligned} \langle \delta(u), \kappa_{\text{nr}} \rangle_{\mathfrak{p}} &= -\text{ord}_{\mathfrak{P}}(u), \\ \langle \delta(u), \kappa_{\text{cyc}} \rangle_{\mathfrak{p}} &= \mathbf{L}_{\mathfrak{P}}(u). \end{aligned}$$

Combining these, we get the desired result.

3.1.1. Tate's local duality. Let v be a finite place of F , V a finite-dimensional representation of G_{F_v} over E . We have the following perfect pairing, which is called “Tate's local duality.”

$$H^1(G_{F_v}, V) \times H^1(G_{F_v}, \text{Hom}(V, E(1))) \rightarrow H^2(G_{F_v}, E(1)) = E.$$

Putting $V = E(\chi^{-1})$, we get the desired E -linear pairing

$$\langle \cdot, \cdot \rangle_v : H^1(G_{F_v}, E(\chi^{-1})) \times H^1(G_{F_v}, E(1)(\chi)) \rightarrow E.$$

We define the unramified part of $H^1(G_{F_v}, V)$ by

$$H_{\text{nr}}^1(G_{F_v}, V) := \text{Ker}[H^1(G_{F_v}, V) \rightarrow H^1(I_v, V)].$$

Here we denote the inertia group of G_{F_v} by I_v . Then we have

$$\begin{aligned} H_{\text{nr}}^1(G_{F_v}, E(\chi^{-1})) &\cong H^1(G_{F_v}/I_v, (E(\chi^{-1}))^{I_v}), \\ H_{\text{nr}}^1(G_{F_v}, E(1)(\chi)) &= \{u \in H^1(G_{F_v}, E(1)(\chi)) \mid \langle \kappa, u \rangle_v = 0 \ \forall \kappa \in H_{\text{nr}}^1(G_{F_v}, E(\chi^{-1}))\}. \end{aligned}$$

We can calculate the dimension of each space [DDP, Lemma 1.3 and §1.2]:

$$\begin{aligned} \dim_E H^1(G_{F_v}, E(\chi^{-1})) &= \dim_E H^1(G_{F_v}, E(1)(\chi)) \\ &= \begin{cases} [F_v : \mathbb{Q}_p] & \text{if } v|p, \chi|_{G_{F_v}} \neq 1, \\ [F_v : \mathbb{Q}_p] + 1 & \text{if } v|p, \chi|_{G_{F_v}} = 1, \\ 1 & \text{if } v \nmid p\infty, \chi|_{G_{F_v}} = 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

$$H_{\text{nr}}^1(G_{F_v}, E(\chi^{-1})) = \begin{cases} E \cdot \kappa_{\text{nr}} & \text{if } \chi|_{G_{F_v}} = 1, \\ \{0\} & \text{otherwise,} \end{cases}$$

$$H_{\text{nr}}^1(G_{F_v}, E(1)(\chi)) \cong \begin{cases} \mathcal{O}_{F_v}^\times \widehat{\otimes} E & \text{if } \chi|_{G_{F_v}} = 1, \\ H^1(G_{F_v}, E(1)(\chi)) & \text{otherwise.} \end{cases}$$

Here we write the completed tensor product by $*\widehat{\otimes} E := (\lim_{\leftarrow} * \otimes_{\mathbb{Z}} \mathbb{Z}/p^n \mathbb{Z}) \otimes_{\mathbb{Z}_p} E$. In particular, we have

$$(3.2) \quad H_{\text{nr}}^1(G_{F_v}, E(\chi^{-1})) = H^1(G_{F_v}, E(\chi^{-1})) \quad \text{if } v \nmid p,$$

$$(3.3) \quad H_{\text{nr}}^1(G_{F_v}, E(1)(\chi)) = H^1(G_{F_v}, E(1)(\chi)) \quad \text{if } \chi|_{G_{F_v}} \neq 1.$$

3.1.2. Kummer Theory. The connecting homomorphism of “Kummer theory” gives an isomorphism

$$H^\times / (H^\times)^n \cong H^1(G_H, \mu_n).$$

Therefore we get $H^\times \widehat{\otimes} E \cong H^1(G_H, E(1))$. Moreover we have an isomorphism

$$\delta: (H^\times \widehat{\otimes} E)^{\chi^{-1}} \cong H^1(G_F, E(1)(\chi))$$

by

$$(H^\times \widehat{\otimes} E)^{\chi^{-1}} \cong H^1(G_H, E(1))^{\chi^{-1}} \cong H^1(G_H, E(1)(\chi))^G \cong H^1(G_F, E(1)(\chi)).$$

Here we used the exact sequence

$$H^1(G, E(1)(\chi)^{G_H}) \rightarrow H^1(G_F, E(1)(\chi)) \xrightarrow{\text{res}} H^1(G_H, E(1)(\chi))^G \rightarrow H^2(G, E(1)(\chi)^{G_H})$$

and $E(1)(\chi)^{G_H} = \{0\}$. For the local field, we similarly get

$$\begin{aligned} H_w^\times \widehat{\otimes} E &\cong H^1(G_{H_w}, E(1)), \\ (H_w^\times \widehat{\otimes} E)^{\chi^{-1}} &\cong H^1(G_{F_v}, E(1)(\chi)) \\ \cup &\quad \cup \\ (\mathcal{O}_{H_w}^\times \widehat{\otimes} E)^{\chi^{-1}} &\cong H_{\text{nr}}^1(G_{F_v}, E(1)(\chi)). \end{aligned}$$

Therefore if we put $H_{\mathfrak{p}}^1(G_F, E(1)(\chi)) := \delta(U_\chi)$, then we can write

$$\begin{aligned} H_{\mathfrak{p}}^1(G_F, E(1)(\chi)) \\ = \{u \in H^1(G_F, E(1)(\chi)) \mid \text{res}_v(u) \in H_{\text{nr}}^1(G_{F_v}, E(1)(\chi)) \ (\forall v \notin S)\}. \end{aligned}$$

Moreover we see that by (3.3)

$$\begin{aligned} H_{\mathfrak{p}}^1(G_F, E(1)(\chi)) \\ = \{u \in H^1(G_F, E(1)(\chi)) \mid \text{res}_v(u) \in H_{\text{nr}}^1(G_{F_v}, E(1)(\chi)) \ (\forall v \neq \mathfrak{p})\}. \end{aligned}$$

(For detail, see [DDP, Proposition 1.4].) Now we put

$$(3.4) \quad \begin{aligned} & H_{\mathfrak{p}, \text{cyc}}^1(G_F, E(\chi^{-1})) \\ & := \text{res}_{\mathfrak{p}}^{-1}(E \cdot \kappa_{\text{nr}} \oplus E \cdot \kappa_{\text{cyc}}) \cap \left(\bigcap_{v \neq \mathfrak{p}} \text{res}_v^{-1}(H_{\text{nr}}^1(G_{F_v}, E(\chi^{-1}))) \right). \end{aligned}$$

Then we have

$$\langle \text{res}_v(\kappa), \text{res}_v(\delta(u)) \rangle_v = 0 \text{ for } v \neq \mathfrak{p}, \kappa \in H_{\mathfrak{p}, \text{cyc}}^1(G_F, E(\chi^{-1})), u \in U_{\chi}.$$

Note that by (3.2) we can write

$$(3.5) \quad \begin{aligned} & H_{\mathfrak{p}, \text{cyc}}^1(G_F, E(\chi^{-1})) \\ & = \text{res}_{\mathfrak{p}}^{-1}(E \cdot \kappa_{\text{nr}} \oplus E \cdot \kappa_{\text{cyc}}) \cap \left(\bigcap_{v \in S_p - \{\mathfrak{p}\}} \text{res}_v^{-1}(H_{\text{nr}}^1(G_{F_v}, E(\chi^{-1}))) \right). \end{aligned}$$

Furthermore, by using Poitou-Tate exact sequence, we can show the following properties [DDP, Lemma 1.5]

$$\begin{aligned} \dim_E H_{\mathfrak{p}, \text{cyc}}^1(G_F, E(\chi^{-1})) &= 1, \\ \text{res}_{\mathfrak{p}}: H_{\mathfrak{p}, \text{cyc}}^1(G_F, E(\chi^{-1})) &\hookrightarrow E \cdot \kappa_{\text{nr}} \oplus E \cdot \kappa_{\text{cyc}}. \end{aligned}$$

§ 3.2. A very rough sketch of the proof of the main theorem.

We reduced the problem to the construction of a cocycle in the previous subsection. Before we go into details we shall give a sketch of the construction in [DDP]. We put

$$\begin{aligned} \mathfrak{n}_R &:= \text{lcm}(\mathfrak{n}, \prod_{\mathfrak{p} \neq \mathfrak{q} \in S_p} \mathfrak{q}), \\ \mathfrak{n}_S &:= \text{lcm}(\mathfrak{n}, \prod_{\mathfrak{q} \in S_p} \mathfrak{q}), \end{aligned}$$

and denote the character modulo \mathfrak{n}_R (resp. \mathfrak{n}_S) associated to χ by χ_R (resp. χ_S).

1. We denote the Eisenstein series of weight k associated to characters η, ψ by $E_k(\eta, \psi)$. $E_k(\eta, \psi)$ is characterized by its \mathfrak{m} -th Fourier coefficients $c(\mathfrak{m}, E_k(\eta, \psi))$, that is

$$c(\mathfrak{m}, E_k(\eta, \psi)) = \sum_{\mathfrak{r} | \mathfrak{m}} \eta(\mathfrak{m}/\mathfrak{r}) \psi(\mathfrak{r}) N\mathfrak{r}^{k-1}$$

for all non-zero integral ideals \mathfrak{m} of \mathcal{O}_F . Here \mathfrak{r} runs over all integral ideals dividing \mathfrak{m} . Consider the product of Eisenstein series

$$P_k := E_1(1, \chi_R) \cdot \frac{2^n}{L(2-k, \omega^{1-k})} E_{k-1}(1, \omega^{1-k}).$$

Then we see that P_k is a Hilbert modular form of weight k , level \mathfrak{n}_S , character $\chi\omega^{1-k}$.

2. By well-known results for (a family of) Eisenstein series, we see that the family $\{P_k\}_k$ becomes a Λ -adic Hilbert modular form. Since we have to apply Wiles' results on ordinary Λ -adic cusp forms, we shall modify the family $\{P_k\}_k$ as in §3.3 in order to get a family of ordinary cusp forms. First we take the ordinary part eP_k by Hida's ordinary operator e . By a general theory of Hilbert modular forms, we can uniquely write eP_k as

$$eP_k = \text{"an ordinary cusp form"} + \sum_{j \in J} a_k(\eta_j, \psi_j) E_k(\eta_j, \psi_j),$$

where J is a finite set of indices, $a_k(\eta_j, \psi_j)$ are constants, $E_k(\eta_j, \psi_j)$ are ordinary Eisenstein series. We can remove some of $\{a_k(\eta_j, \psi_j) E_k(\eta_j, \psi_j)\}_{j \in J}$ by multiplying eP_k by $T_j - \alpha_j$ with T_j a suitable Hecke operator, α_j the eigenvalue for T_j , $E_k(\eta_j, \psi_j)$ (for detail, see [DDP, Lemma 2.9] or Lemma 3.9 in this paper). Here each operator T_j has to satisfy a certain condition (that is, $(T_j - \alpha_j)E_1(1, \chi_S) \neq 0$) in order to guarantee that the weight 1 specialization of the ordinary Λ -adic cusp form obtained finally is non-zero (that is, $\nu_1(\mathcal{F}) \neq 0$ in the next step). When we can not remove $a_k(\eta_j, \psi_j) E_k(\eta_j, \psi_j)$ by Hecke operators for this reason, we compute $a_k(\eta_j, \psi_j)$ explicitly by comparing constant terms ([DDP, Propositions 2.6, 2.7] or Proposition 3.7 in this paper) and subtract $a_k(\eta_j, \psi_j) E_k(\eta_j, \psi_j)$ from eP_k . By a combination of these methods, we get a family of ordinary cusp forms $\{\mathcal{F}_k\}_k$ ([DDP, Corollary 2.10] or Corollary 3.10 in this paper). We see that this family $\{\mathcal{F}_k\}_k$ becomes an ordinary Λ -adic cusp form, which is denoted by \mathcal{F} .

3. Let Λ be the Iwasawa algebra, ν_k the weight k specialization $\Lambda \rightarrow \mathcal{O}_E$, $\Lambda_{(k)}$ the localization of Λ at $\text{Ker } \nu_k$. Then under the assumption (3.1), we see that Fourier coefficients of \mathcal{F} are in $\Lambda_{(1)}$ and its weight 1 specialization is equal to $E_1(1, \chi_S)$ up to multiplication by a non-zero constant. More precisely, in Proposition 3.13, we write \mathcal{F} in the form of

$$\mathcal{F} = \text{"a } \Lambda\text{-adic Hecke operator"} \cdot (u\mathcal{E}(1, \chi) + v\mathcal{E}(\chi, 1) + \omega\mathcal{P}^0).$$

Here $\mathcal{E}(1, \chi), \mathcal{E}(\chi, 1)$ are Λ -adic Eisenstein series whose Fourier coefficients are in Λ , \mathcal{P}^0 is the ordinary Λ -adic Hilbert modular form with respect to $\{eP_k\}_k$, and u, v, w are functions expressed in terms of $\mathcal{L}_{\text{an}}(s, \chi)$, $\mathcal{L}_{\text{an}}(s, \chi^{-1})$ as in Corollary 3.10. We need the additional condition of (3.1) in the case of $|S_p| = 1$ in order to ensure that u, v, w are elements in $\Lambda_{(1)}$. Moreover Leopoldt's conjecture implies that $\nu_1(\mathcal{F}) = t \cdot E_1(1, \chi_S)$ with $t \in E^\times$. We note that $\nu_1(\mathcal{F})$ is an eigenform, but \mathcal{F} itself is not. Therefore we may guess \mathcal{F} is "approximately" an eigenform near $k = 1$. In fact, by a direct computation, we see that $\mathcal{F} \bmod (\text{Ker } \nu_1)^2$ is a simultaneous eigenform of all Hecke operators whose eigen values are contained

in $\Lambda_{(1)}/(\text{Ker } \nu_1)^2$. To explain more precisely, let T_l be the l -th Hecke operator, $c(\mathfrak{m}, \mathcal{F}) \in \Lambda_{(1)}$ the \mathfrak{m} -th Fourier coefficient of \mathcal{F} . Then we will see that

$$c(\mathfrak{m}, T_l \mathcal{F}) \equiv \alpha_l c(\mathfrak{m}, \mathcal{F}) \pmod{(\text{Ker } \nu_1)^2}$$

with $\alpha_l \in \Lambda_{(1)}$. (Strictly speaking, Fourier coefficients of $\mathcal{H} := u\mathcal{E}(1, \chi) + v\mathcal{E}(\chi, 1) + \omega\mathcal{P}^0$ are computed in §3.5, instead of those of \mathcal{F} .) Therefore we get a Λ -algebra homomorphism

$$\phi_{1+\varepsilon} : \mathbf{T} \rightarrow \Lambda_{(1)}/(\text{Ker } \nu_1)^2, \quad T_l \mapsto \alpha_l \pmod{(\text{Ker } \nu_1)^2}.$$

Here we denote by \mathbf{T} the ordinary Λ -adic Hecke algebra acting on the space of ordinary Λ -adic cusp forms.

4. Define

$$\begin{aligned} \phi_1 : \mathbf{T} &\rightarrow \Lambda_{(1)}/\text{Ker } \nu_1 \cong E, \\ \phi_1(T) &:= \phi_{1+\varepsilon}(T) \pmod{\text{Ker } \nu_1}. \end{aligned}$$

We denote the localization of \mathbf{T} at ϕ_1 by $\mathbf{T}_{(1)}$, the total ring of fractions of $\mathbf{T}_{(1)}$ by $\mathcal{F}_{\mathbf{T}_{(1)}}$. Wiles constructed a “big” Galois representation (§3.4)

$$\rho_{(1)} : G_F \rightarrow \text{GL}(\mathcal{F}_{\mathbf{T}_{(1)}})$$

which is characterized by

$$\text{Tr } \rho_{(1)}(\text{Frob}_l) = \text{the } l\text{-th Hecke operator } T_l$$

for almost all primes l . Taking a suitable basis, we write $\rho_{(1)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then we can show that $\phi_{1+\varepsilon} \circ a, \phi_{1+\varepsilon} \circ d$ become characters $\psi_1, \psi_2 : G_F \rightarrow (\Lambda_{(1)}/(\text{Ker } \nu_1)^2)^\times$ which are defined in Definition 3.16. Therefore $\phi_{1+\varepsilon} \circ (b/d)$ becomes a cocycle

$$K : G_F \rightarrow \Lambda_{(1)}/(\text{Ker } \nu_1)^2.$$

5. We may regard any element $f \in \Lambda_{(1)}$ as a meromorphic function $f(s)$ on \mathbb{Z}_p which is analytic at $s = 1$. Then we can identify

$$\begin{aligned} \Lambda_{(1)}/(\text{Ker } \nu_1)^2 &\cong E[\varepsilon]/(\varepsilon^2) = E \oplus E \cdot \varepsilon, \\ f &\longleftrightarrow f(1) + f'(1)\varepsilon, \end{aligned}$$

where $E[\varepsilon]$ is the polynomial ring in one variable ε . After multiplication by a constant, we may assume that

$$\{0\} \neq K(G_F) \subset E \cdot \varepsilon.$$

Then we can define a function

$$\kappa: G_F \rightarrow E$$

by

$$K(\sigma) = \kappa(\sigma) \cdot \varepsilon \quad (\sigma \in G_F),$$

which again becomes a cocycle. In §3.6, we will check this κ satisfies the desired conditions as in Conjecture 3.3.

Remark. As stated in [DDP, *Construction of a cusp form*, p445], some techniques in [Ri] and [Wi] are used to construct the cocycle κ . For example, Wiles [Wi, proof of Theorem 4.1, p508] suggested the following strategy: Let \mathbf{T} be the Λ -adic Hecke algebra acting on the space of ordinary Λ -adic cusp forms, \mathcal{F} an ordinary Λ -adic cusp form, and \mathfrak{b} an ideal of Λ . Assume that \mathcal{F} is a Hecke eigenform mod \mathfrak{b} , that is, there exist elements $\alpha_{\mathfrak{l}} \in \Lambda$ satisfying $c(\mathfrak{m}, T_{\mathfrak{l}}\mathcal{F}) \equiv \alpha_{\mathfrak{l}}c(\mathfrak{m}, \mathcal{F}) \pmod{\mathfrak{b}}$ for all $\mathfrak{l}, \mathfrak{m}$. Then we get a homomorphism $\eta_{\mathcal{F}}: \mathbf{T} \rightarrow \Lambda/\mathfrak{a}$ given by $T_{\mathfrak{l}} \mapsto \alpha_{\mathfrak{l}}$, where $\mathfrak{a} := \{\lambda \in \Lambda \mid \lambda c(\mathcal{O}_F, \mathcal{F}) \in \mathfrak{b}\}$. In [Wi], such an ordinary Λ -adic cusp form \mathcal{F}' was constructed by modifying a product of Eisenstein series. This “strategy” and a similar modification of a product of Eisenstein series are used also in [DDP]. However, taking $\mathfrak{a} = (\text{Ker } \nu_1)^2$ seems to be one of their new ideas, in order to relate the homomorphism $\eta_{\mathcal{F}}$ ($= \phi_{1+\varepsilon}$ in the above sketch) to the first derivatives of p -adic L -function. Moreover we need an explicit formula [DDP, (94)] ((3.13) in this paper) for Fourier coefficients in order to investigate the cocycle κ . By this formula, we easily see that \mathcal{H} is (and hence \mathcal{F} is also) a Hecke eigenform mod $(\text{Ker } \nu_1)^2$ with eigenvalues given by [DDP, Proposition 3.6] (Proposition 3.17 in this paper), and we can write down the map $\phi_{1+\varepsilon}$ explicitly as in [DDP, Theorem 3.7] (Theorem 3.18 in this paper). We need this expression of $\phi_{1+\varepsilon}$ to check that the cocycle κ , which is defined by using $\phi_{1+\varepsilon}$ as in (3.15), satisfies the desired formulas.

§ 3.3. Λ -adic Hilbert modular forms.

We will use the following notations. F is a totally real field of degree n . The narrow ideal class group of F is denoted by $\text{Cl}^+(F)$. For each class $\lambda \in \text{Cl}^+(F)$, we fix a representative $\mathfrak{t}_{\lambda} \in \lambda$. We put $M_k(\mathfrak{n}, \psi)$ (resp. $S_k(\mathfrak{n}, \psi)$) to be the space of Hilbert modular forms (resp. Hilbert cusp forms) over F , of weight k , level \mathfrak{n} , character ψ . For $f \in M_k(\mathfrak{n}, \psi)$, we denote the normalized Fourier coefficient by $c_{\lambda}(0, f)$ (resp. $c(\mathfrak{m}, f)$) at $\lambda \in \text{Cl}^+(F)$ (resp. at a non-zero integral ideal $\mathfrak{m} \subset \mathcal{O}_F$). Let $\Lambda \cong \mathcal{O}_E[[T]]$ be the Iwasawa algebra equipped with the weight k specialization $\nu_k: \Lambda \rightarrow \mathcal{O}_E$ for $k \in \mathbb{Z}_p$. We denote the fraction field of Λ by \mathcal{F}_{Λ} , the localization of Λ at $\text{Ker } \nu_k$ by $\Lambda_{(k)}$. Then ν_k is extended to the homomorphism $\nu_k: \Lambda_{(k)} \rightarrow E$.

Definition 3.4. A family $\mathcal{F} = \{c(\mathfrak{m}, \mathcal{F}), c_\lambda(0, \mathcal{F}) \mid \mathfrak{m}, \lambda\}$ of formal Fourier coefficients $c(\mathfrak{m}, \mathcal{F}), c_\lambda(0, \mathcal{F}) \in \Lambda$ is called a Λ -adic form (resp. a Λ -adic cusp form) of level \mathfrak{n} , character χ if it satisfies

for almost all $k \geq 2$, there exist $f_k \in M_k(\mathfrak{n}', \chi\omega^{1-k})$ (resp. $S_k(\mathfrak{n}', \chi\omega^{1-k})$) satisfying

$$\nu_k(c(\mathfrak{m}, \mathcal{F})) = c(\mathfrak{m}, f_k), \nu_k(c_\lambda(0, \mathcal{F})) = c_\lambda(0, f_k) \quad (\forall \lambda, \mathfrak{m}).$$

Here we put $\mathfrak{n}' := \text{lcm}(\mathfrak{n}, \prod_{q \in S_p} q)$. For such an \mathcal{F} , we write $\mathcal{F}_k := \nu_k(\mathcal{F}) := f_k$. We denote the space of all Λ -adic forms (resp. Λ -adic cusp forms) of level \mathfrak{n} , character χ by $\mathcal{M}(\mathfrak{n}, \chi)$ (resp. $\mathcal{S}(\mathfrak{n}, \chi)$). Actually we also call an element in $\mathcal{M}(\mathfrak{n}, \chi) \otimes_\Lambda \mathcal{F}_\Lambda$ (resp. $\mathcal{S}(\mathfrak{n}, \chi) \otimes_\Lambda \mathcal{F}_\Lambda$) a Λ -adic form (resp. a Λ -adic cusp form).

We now recall some properties of the Eisenstein series $E_k(\eta, \psi)$, which is one of the main tools. It is constructed explicitly and is characterized by the following property. For details we refer to [Shim1].

Proposition 3.5 ([Shim1, Proposition 3.4]). *Let η, ψ be characters of the narrow ideal class groups modulo $\mathfrak{m}_\eta, \mathfrak{m}_\psi$ with associated signs $q, r \in (\mathbb{Z}/2\mathbb{Z})^n$ respectively. Assume that $q + r \equiv (k, k, \dots, k) \pmod{2\mathbb{Z}^n}$ with $k \in \mathbb{N}$. Then there exists an element $E_k(\eta, \psi) \in M_k(\mathfrak{m}_\eta \mathfrak{m}_\psi, \eta\psi)$ satisfying*

$$(3.6) \quad c(\mathfrak{m}, E_k(\eta, \psi)) = \sum_{\mathfrak{r} \mid \mathfrak{m}} \eta(\mathfrak{m}/\mathfrak{r}) \psi(\mathfrak{r}) N\mathfrak{r}^{k-1}.$$

Here \mathfrak{r} runs over all integral ideals dividing \mathfrak{m} .

Proof. We only give a sketch of the construction of the desired modular form in [Shim1, §3]. We denote the upper half plane by $\mathfrak{H} := \{z \in \mathbb{C} \mid \text{Im } z > 0\}$, the set of all embeddings $F \hookrightarrow \mathbb{R}$ by $\{\tau_1, \dots, \tau_n\}$. Let $\mathfrak{a}, \mathfrak{b}$ be fractional ideals of F , $a_0, b_0 \in F$, $k \in \mathbb{N}$, and U a subgroup of \mathcal{O}_F^\times of finite index. We assume that U is “sufficiently small”, that is, $N(u)^k = 1$, $ua_0 - a_0 \in \mathfrak{a}$, $ub_0 - b_0 \in \mathfrak{b}$ for all $u \in U$. Then we define for $z = (z_1, \dots, z_n) \in \mathfrak{H}^n$, $s \in \mathbb{C}$

$$E_{k,U}(z, s; a_0, b_0; \mathfrak{a}, \mathfrak{b}) := D_F^{\frac{1}{2}} N(\mathfrak{b}) (-2\pi i)^{-kn} \sum_{(a,b)U} \prod_{i=1}^n (a^{\tau_i} z_i + b^{\tau_i})^{-k} |a^{\tau_i} z_i + b^{\tau_i}|^{-2s}.$$

Here D_F is the discriminant of F and the element $a^{\tau_i} z_i + b^{\tau_i} \in \mathbb{C}$ is well-defined since $\mathbb{R}, \mathfrak{H} \subset \mathbb{C}$. The sum over $(a,b)U$ runs over all cosets $(a,b)U \in \{(a,b) \in F \times F - \{(0,0)\} \mid a - a_0 \in \mathfrak{a}, b - b_0 \in \mathfrak{b}\} / U$, where the action of $u \in U$ is defined by $(a,b)u := (au, bu)$. By assumption on U , this definition does not depend on the choice of representatives (a,b) . The series is convergent for $\text{Re}(2s + k) > 2$ and continued meromorphically to the whole complex s -plane. Moreover, we can show that

$$E_{k,U}(z; a_0, b_0; \mathfrak{a}, \mathfrak{b}) := E_{k,U}(z, 0; a_0, b_0; \mathfrak{a}, \mathfrak{b})$$

is a Hilbert modular form of weight k in the traditional sense (except when $n = 1$, $k = 2$). We note that when $k > 2$, we can define the function $E_{k,U}(z; a_0, b_0; \mathfrak{a}, \mathfrak{b})$ without introducing another variable s and without analytic continuation. Taking a sufficiently small U , we put

$$\begin{aligned} G(a, b, \mathfrak{x}, \mathfrak{y}) &:= \Gamma(k)^n N(\mathfrak{m}_\psi)^{-1} \sum_{t \in \mathfrak{m}_\psi^{-1} \mathfrak{d}^{-1} \mathfrak{y}^{-1} / \mathfrak{d}^{-1} \mathfrak{y}^{-1}} \mathbf{e}_F(-tb) E_{k,U}(z; a, t; \mathfrak{x}, \mathfrak{d}^{-1} \mathfrak{y}^{-1}), \\ H(\mathfrak{x}, \mathfrak{y}) &:= \sum_{a \in \mathfrak{x} / \mathfrak{x} \mathfrak{m}_\eta, b \in \mathfrak{y} / \mathfrak{y} \mathfrak{m}_\psi} \text{sgn}(a^q) \eta(a \mathfrak{x}^{-1}) \text{sgn}(b^r) \psi(b \mathfrak{y}^{-1}) N(\mathfrak{y})^{1-k} G(a, b; \mathfrak{x} \mathfrak{m}_\eta, \mathfrak{y}), \\ K(\mathfrak{t}) &:= [\mathcal{O}_F^\times : U]^{-1} \sum_{\mathfrak{x}} H(\mathfrak{x}, \mathfrak{t} \mathfrak{x}^{-1}) \end{aligned}$$

for fractional ideals $\mathfrak{x}, \mathfrak{y}, \mathfrak{t}$, elements $a \in F$, $b \in \mathfrak{y}$. Here \mathfrak{d} is the different of F , $\mathbf{e}_F(x) := e^{2\pi i \cdot \text{tr}_{F/\mathbb{Q}}(x)}$, $\text{sgn}(x^{(m_1, \dots, m_n)}) := \prod_{i=1}^n \text{sgn}((x^{\tau_i})^{m_i})$ ($x \in F$, $m_1, \dots, m_n \in \mathbb{Z}$), and \mathfrak{x} runs over a complete set of representatives for the ideal class group of F . Note that $K(\mathfrak{t})$ does not depend on the choice of U when U is sufficiently small. Then we can show that

$$E_k(\eta, \psi) := (g_\lambda)_{\lambda \in \text{Cl}^+(F)} \text{ with } g_\lambda = N(\mathfrak{t}_\lambda)^{\frac{k}{2}} K(\mathfrak{t}_\lambda)$$

satisfies the desired properties. We note that (3.6) is equivalent to [Shim1, (3.21)] as mentioned in [Shim1, p24, 118]. \square

As commented in [DDP, Remark 2.2], the explicit formula for the constant terms of $E_k(\eta, \psi)$ seem to be well-known to the experts (e.g., in [Ka]). Since notations differ, these are recalculated in [DDP, Propositions 2.1, 2.3, 2.4, 2.5]. For example, we have

$$c_\lambda(0, E_k(\eta, \psi)) = \begin{cases} 2^{-n} \eta^{-1}(\mathfrak{t}_\lambda) L(1-k, \psi \eta^{-1}) & \text{if } k > 1, \mathfrak{m}_\eta = (1), \\ 0 & \text{if } k > 1, \mathfrak{m}_\eta \neq (1), \\ 2^{-n} \eta^{-1}(\mathfrak{t}_\lambda) L(0, \psi \eta^{-1}) & \text{if } k = 1, \mathfrak{m}_\eta = (1), \mathfrak{m}_\psi \neq (1), \\ 2^{-n} \psi^{-1}(\mathfrak{t}_\lambda) L(0, \eta \psi^{-1}) & \text{if } k = 1, \mathfrak{m}_\eta \neq (1), \mathfrak{m}_\psi = (1), \\ 2^{-n} \eta^{-1}(\mathfrak{t}_\lambda) L(0, \psi \eta^{-1}) + 2^{-n} \psi^{-1}(\mathfrak{t}_\lambda) L(0, \eta \psi^{-1}) & \text{if } k = 1, \mathfrak{m}_\eta = (1), \mathfrak{m}_\psi = (1), \\ 0 & \text{if } k = 1, \mathfrak{m}_\eta \neq (1), \mathfrak{m}_\psi \neq (1). \end{cases}$$

Definition 3.6. Additionally assume $L(1-k, \psi) \neq 0$. Then we define the normalized Eisenstein series $G_k(1, \psi)$ by

$$G_k(1, \psi) := \frac{2^n}{L(1-k, \psi)} E_k(1, \psi).$$

Let χ be a primitive character with conductor \mathfrak{n} , as in the setting of the Gross-Stark conjecture. We put

$$\begin{aligned} \mathfrak{n}_S &:= \text{lcm} \left(\mathfrak{n}, \prod_{\mathfrak{q} \in S_p} \mathfrak{q} \right) = \mathfrak{n} \cdot \prod_{\mathfrak{q} \in S_p, \mathfrak{q} \nmid \mathfrak{n}} \mathfrak{q}, \\ \mathfrak{n}_R &:= \text{lcm} \left(\mathfrak{n}, \prod_{\mathfrak{p} \neq \mathfrak{q} \in S_p} \mathfrak{q} \right) = \mathfrak{n} \cdot \prod_{\mathfrak{p} \neq \mathfrak{q} \in S_p, \mathfrak{q} \nmid \mathfrak{n}} \mathfrak{q}, \end{aligned}$$

and denote the character modulo \mathfrak{n}_S (resp. \mathfrak{n}_R) associated to χ by χ_S (resp. χ_R). We consider the following product of Eisenstein series.

$$P_k := E_1(1, \chi_R) \cdot G_{k-1}(1, \omega^{1-k}) \in M_k(\mathfrak{n}_S, \chi \omega^{1-k}).$$

Here we consider ω^{1-k} (resp. $\chi \omega^{1-k}$) is a character modulo $\prod_{\mathfrak{q} \in S_p} \mathfrak{q}$ (resp. \mathfrak{n}_S) for any k (even if $k = 1$). We denote by $E_k(\mathfrak{n}, \psi)$ the Eisenstein series part of $M_k(\mathfrak{n}, \psi)$, which is the \mathbb{C} -subspace of $M_k(\mathfrak{n}, \psi)$ spanned by Eisenstein series in $M_k(\mathfrak{n}, \psi)$. Then we have the following decomposition which is stable under the action of Hecke operators:

$$M_k(\mathfrak{n}, \psi) = S_k(\mathfrak{n}, \psi) \oplus E_k(\mathfrak{n}, \psi).$$

For details we refer the reader to [Shim2, §7, §8]. Therefore we can uniquely write P_k as

$$P_k = \text{“A cusp form”} + \sum_{(\eta, \psi) \in J} a_k(\eta, \psi) E_k(\eta, \psi)$$

with constants $a_k(\eta, \psi) \in \mathbb{C}$. Here we put

$$J := \{(\eta, \psi) \mid \eta, \psi \text{ characters of modulus } \mathfrak{m}_\eta, \mathfrak{m}_\psi \text{ with } \mathfrak{m}_\eta \mathfrak{m}_\psi = \mathfrak{n}_S, \eta\psi = \chi \omega^{1-k}\}.$$

We will remove the Eisenstein series part of P_k (strictly speaking, the Eisenstein series part of the ordinary part P_k^o of P_k) in this expression to get a family of ordinary cusp forms $\mathcal{F} = \{F_k\}_k$ satisfying $F_1 \neq 0$ as follows. We compute the eigenvalue α of $E_k(\eta, \psi)$ with respect to a suitable Hecke operator T and multiply P_k by $T - \alpha$ for each pair $(\eta, \psi) \in J$. It can be done when $\eta \neq 1$, $|S_p| > 1$ or $\eta \neq 1, \chi$, $|S_p| = 1$ by Lemma 3.9. When $\eta = 1$, $|S_p| > 1$ or $\eta = 1, \chi$, $|S_p| = 1$, we have to compute $a_k(\eta, \psi)$ explicitly as in Proposition 3.7 and subtract $a_k(\eta, \psi) E_k(\eta, \psi)$ from P_k .

First of all, we get the following proposition [DDP, Proposition 2.6, 2.7] by comparing their constant terms.

Proposition 3.7. *For $k \geq 2$, we have*

$$a_k(1, \chi \omega^{1-k}) = -\mathcal{L}_{\text{an}}(k, \chi)^{-1}.$$

If $k > 2$, $|S_p| = 1$ then we have

$$a_k(\chi, \omega^{1-k}) = -\mathcal{L}_{\text{an}}(k, \chi^{-1})^{-1} \cdot \langle N\mathfrak{n} \rangle^{k-1}.$$

Here we put $\langle z \rangle := z/\omega(z)$ for $z \in \mathbb{Z}_p^\times$ as usual.

We denote the \mathfrak{l} -th Hecke operators by $T_{\mathfrak{l}}, U_{\mathfrak{l}}$ for prime ideals \mathfrak{l} of \mathcal{O}_F . Then for any pair $(\eta, \psi) \in J$, we can easily see that

$$\begin{aligned} T_{\mathfrak{l}}E_k(\eta, \psi) &= (\eta(\mathfrak{l}) + \psi(\mathfrak{l})N\mathfrak{l}^{k-1})E_k(\eta, \psi) \quad (\mathfrak{l} \nmid \mathfrak{n}_S), \\ U_{\mathfrak{l}}E_k(\eta, \psi) &= (\eta(\mathfrak{l}) + \psi(\mathfrak{l})N\mathfrak{l}^{k-1})E_k(\eta, \psi) \quad (\mathfrak{l} \mid \mathfrak{n}_S) \\ &= \begin{cases} \eta(\mathfrak{l})E_k(\eta, \psi) & (\mathfrak{l} \mid \mathfrak{m}_{\psi}), \\ \psi(\mathfrak{l})N\mathfrak{l}^{k-1}E_k(\eta, \psi) & (\mathfrak{l} \mid \mathfrak{m}_{\eta}). \end{cases} \end{aligned}$$

Therefore we get the following Proposition and Lemma [DDP, Proposition 2.8, Lemma 2.9].

Proposition 3.8. *For simplicity, we fix an embedding $\overline{\mathbb{Q}_p} \subset \mathbb{C}$. For any subring A of $\overline{\mathbb{Q}_p}$, we denote by $M_k(\mathfrak{n}_S, \chi\omega^{1-k}; A)$ the space of modular forms whose Fourier coefficients are in A . Let e be the ordinary operator defined by*

$$e := \lim_{r \rightarrow \infty} \left(\prod_{\mathfrak{q} \in S_p} U_{\mathfrak{q}} \right)^{r!}$$

on $M_k(\mathfrak{n}_S, \chi\omega^{1-k}; \mathcal{O}_L)$ with L a finite extension of \mathbb{Q}_p . It can be extended to an operator on $M_k(\mathfrak{n}_S, \chi\omega^{1-k}; L)$ linearly. Then we have for $k \geq 2$

$$eE_k(\eta, \psi) = \begin{cases} E_k(\eta, \psi) & \text{if } (p, \mathfrak{m}_{\eta}) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore we can write

$$P_k^{\circ} := eP_k = \text{“An ordinary cusp form”} + \sum_{(\eta, \psi) \in J^{\circ}} a_k(\eta, \psi)E_k(\eta, \psi)$$

with $J^{\circ} := \{(\eta, \psi) \in J \mid (p, \mathfrak{m}_{\eta}) = 1\}$.

Lemma 3.9 ([DDP, Lemma 2.9]). *For $(\eta, \psi) \in J^{\circ}$ with $\eta \neq 1, \chi$, there exists a prime ideal $\mathfrak{l} = \mathfrak{l}_{\eta, \psi} \nmid \mathfrak{n}_S$ satisfying*

$$\begin{aligned} T_{\eta, \psi, k}E_k(\eta, \psi) &= 0, \\ T_{\eta, \psi, k}E_1(1, \chi_S) &\neq 0 \\ \text{with } T_{\eta, \psi, k} &:= T_{\mathfrak{l}} - \eta(\mathfrak{l}) - \psi(\mathfrak{l})N\mathfrak{l}^{k-1}. \end{aligned}$$

Additionally assume that $|S_p| > 1$. Then for $\mathfrak{q} \in S_p - \{\mathfrak{p}\}$, we have

$$\begin{aligned} T_{\chi, \omega^{1-k}, k} E_k(\chi, \omega^{1-k}) &= 0, \\ T_{\chi, \omega^{1-k}, k} E_1(1, \chi_S) &\neq 0 \\ \text{with } T_{\chi, \omega^{1-k}, k} &:= U_{\mathfrak{q}} - \chi(\mathfrak{q}). \end{aligned}$$

Summarizing the above, we get the following result.

Corollary 3.10 ([DDP, Corollary 2.10]). *Put*

$$\begin{aligned} u_k &:= \begin{cases} \frac{1}{1 + \mathcal{L}_{\text{an}}(k, \chi)} & \text{if } |S_p| > 1, \\ \frac{\mathcal{L}_{\text{an}}(k, \chi)^{-1}}{\mathcal{L}_{\text{an}}(k, \chi)^{-1} + \mathcal{L}_{\text{an}}(k, \chi^{-1})^{-1} \langle N\mathfrak{n} \rangle^{k-1} + 1} & \text{if } |S_p| = 1, \end{cases} \\ v_k &:= \begin{cases} 0 & \text{if } |S_p| > 1, \\ \frac{\mathcal{L}_{\text{an}}(k, \chi^{-1})^{-1} \langle N\mathfrak{n} \rangle^{k-1}}{\mathcal{L}_{\text{an}}(k, \chi)^{-1} + \mathcal{L}_{\text{an}}(k, \chi^{-1})^{-1} \langle N\mathfrak{n} \rangle^{k-1} + 1} & \text{if } |S_p| = 1, \end{cases} \\ w_k &:= \begin{cases} \frac{\mathcal{L}_{\text{an}}(k, \chi)}{1 + \mathcal{L}_{\text{an}}(k, \chi)} & \text{if } |S_p| > 1, \\ \frac{1}{\mathcal{L}_{\text{an}}(k, \chi)^{-1} + \mathcal{L}_{\text{an}}(k, \chi^{-1})^{-1} \langle N\mathfrak{n} \rangle^{k-1} + 1} & \text{if } |S_p| = 1, \end{cases} \\ H_k &:= u_k E_k(1, \chi \omega^{1-k}) + v_k E_k(\chi, \omega^{1-k}) + w_k P_k^o, \\ F_k &:= \begin{cases} \left(\prod_{(\eta, \psi) \in J^o, \eta \neq 1} T_{\eta, \psi, k} \right) H_k & \text{if } |S_p| > 1, \\ \left(\prod_{(\eta, \psi) \in J^o, \eta \neq 1, \chi} T_{\eta, \psi, k} \right) H_k & \text{if } |S_p| = 1. \end{cases} \end{aligned}$$

Then $F_k \in S_k(\mathfrak{n}_S, \chi \omega^{1-k})$.

Hereafter in this subsection, we see the family $\{F_k\}_k$ of Hilbert cusp forms becomes a Λ -adic cusp form. We can define the \mathfrak{l} -th Hecke operators $T_{\mathfrak{l}}, U_{\mathfrak{l}}$ on spaces $\mathcal{M}(\mathfrak{n}, \chi)$, $\mathcal{S}(\mathfrak{n}, \chi)$ as usual. Then the ordinary parts of the spaces of Λ -adic forms is defined by

$$\begin{aligned} \mathcal{M}^o(\mathfrak{n}, \chi) &:= e\mathcal{M}(\mathfrak{n}, \chi), \\ \mathcal{S}^o(\mathfrak{n}, \chi) &:= e\mathcal{S}(\mathfrak{n}, \chi) \end{aligned}$$

with $e := \lim_{r \rightarrow \infty} \left(\prod_{\mathfrak{q} \in S_p} U_{\mathfrak{q}} \right)^{r!}$. We consider the following Λ -algebras of Hecke operators:

$$\begin{aligned} \tilde{\mathbf{T}} &\subset \text{End}_{\Lambda}(\mathcal{M}^o(\mathfrak{n}, \chi)), \\ \mathbf{T} &\subset \text{End}_{\Lambda}(\mathcal{S}^o(\mathfrak{n}, \chi)) \end{aligned}$$

generated by $T_{\mathfrak{l}}, U_{\mathfrak{l}}$ over Λ . The following is a well-known fact.

Proposition 3.11. *If $\eta\psi$ is totally odd, there exists a Λ -adic Hecke eigenform $\mathcal{E}(\eta, \psi) \in \mathcal{M}(\mathfrak{m}_\eta \mathfrak{m}_\psi, \eta\psi) \otimes_\Lambda \mathcal{F}_\Lambda$ satisfying*

$$\nu_k(\mathcal{E}(\eta, \psi)) = E_k(\eta, \psi\omega^{1-k}).$$

We define a “weight shifted Λ -adic form” as follows. For the proof, see [DDP, Proposition 3.3].

Proposition 3.12. *We denote by \mathcal{M}' the space of all families of formal Fourier coefficients $\mathcal{F} = \{c(\mathfrak{m}, \mathcal{F}), c_\lambda(0, \mathcal{F}) \mid \mathfrak{m}, \lambda\}$ ($c(\mathfrak{m}, \mathcal{F}), c_\lambda(0, \mathcal{F}) \in \Lambda$) satisfying*

$$\begin{aligned} &\text{for almost all } k \geq 2, \text{ there exist } \nu_k(\mathcal{F}) \in M_{k-1}(p, \omega^{1-k}) \text{ satisfying} \\ &\nu_k(c(\mathfrak{m}, \mathcal{F})) = c(\mathfrak{m}, \nu_k(\mathcal{F})), \nu_k(c_\lambda(0, \mathcal{F})) = c_\lambda(0, \nu_k(\mathcal{F})) \quad (\forall \lambda, \mathfrak{m}). \end{aligned}$$

Then there exists an element $\mathcal{G} \in \mathcal{M}' \otimes_\Lambda \mathcal{F}_\Lambda$ satisfying

$$\nu_k(\mathcal{G}) := G_{k-1}(1, \omega^{1-k}).$$

Moreover, assuming Leopoldt's conjecture, we have

$$\begin{aligned} \mathcal{G} &\in \mathcal{M}' \otimes_\Lambda \Lambda_{(1)}, \\ \nu_1(\mathcal{G}) &= 1. \end{aligned}$$

Summing up these, we get the following proposition [DDP, Proposition 3.4, Lemma 3.5].

Proposition 3.13. *We denote the p -adic interpolations of $u_k, v_k, w_k, T_{\eta, \psi, k}$ in Lemma 3.9, Corollary 3.10 by $u, v, w, T_{\eta, \psi}$ respectively. We note that the condition $\text{ord}_{s=1}(\mathcal{L}_{\text{an}}(s, \chi) + \mathcal{L}_{\text{an}}(s, \chi^{-1})) = \text{ord}_{s=1} \mathcal{L}_{\text{an}}(s, \chi^{-1})$ in the case of $|S_p| = 1$ assures that $u, v, w \in \Lambda_{(1)}$. Put*

$$\begin{aligned} \mathcal{P} &:= E_1(1, \chi_R) \mathcal{G}, \\ \mathcal{P}^o &:= e\mathcal{P}, \\ \mathcal{H} &:= u\mathcal{E}(1, \chi) + v\mathcal{E}(\chi, 1) + w\mathcal{P}^o, \\ \mathcal{F} &:= \begin{cases} \left(\prod_{(\eta, \psi) \in J^o, \eta \neq 1} T_{\eta, \psi} \right) \mathcal{H} & \text{if } |S_p| > 1, \\ \left(\prod_{(\eta, \psi) \in J^o, \eta \neq 1, \chi} T_{\eta, \psi} \right) \mathcal{H} & \text{if } |S_p| = 1. \end{cases} \end{aligned}$$

Then under the assumption (3.1):

If $|S_p| > 1$, assume that Leopoldt's conjecture is true for F .

If $|S_p| = 1$, assume that Leopoldt's conjecture is true for F ,

and that $\text{ord}_{s=1}(\mathcal{L}_{\text{an}}(s, \chi) + \mathcal{L}_{\text{an}}(s, \chi^{-1})) = \text{ord}_{s=1} \mathcal{L}_{\text{an}}(s, \chi^{-1})$,

we have

$$\begin{aligned}\mathcal{P} &\in \mathcal{M}(\mathfrak{n}, \chi) \otimes_{\Lambda} \Lambda_{(1)}, \\ \mathcal{P}^o, \mathcal{H} &\in \mathcal{M}^o(\mathfrak{n}, \chi) \otimes_{\Lambda} \Lambda_{(1)}, \\ \mathcal{F} &\in \mathcal{S}^o(\mathfrak{n}, \chi) \otimes_{\Lambda} \Lambda_{(1)}.\end{aligned}$$

Moreover we have for all $k \geq 2$

$$\begin{aligned}\nu_k(\mathcal{P}) &= P_k, \\ \nu_k(\mathcal{P}^o) &= P_k^o, \\ \nu_k(\mathcal{H}) &= H_k, \\ \nu_k(\mathcal{F}) &= F_k,\end{aligned}$$

and for $k = 1$

$$\begin{aligned}\nu_1(\mathcal{P}) &= \nu_1(\mathcal{P}^o) = E_1(1, \chi_R), \\ \nu_1(\mathcal{H}) &= E_1(1, \chi_S), \\ \nu_1(\mathcal{F}) &= t \cdot E_1(1, \chi_S)\end{aligned}$$

with an element $t \in E^\times$.

§ 3.4. Wiles' "big Galois representation".

To describe Wiles' result, we use the following notations.

- Define the homomorphism associated to $\nu_1(\mathcal{E}(1, \chi)) = E_1(1, \chi_S)$ by

$$\begin{aligned}\phi_1: \tilde{\mathbf{T}} \otimes_{\Lambda} \Lambda_{(1)} &\rightarrow E, \\ T &\mapsto \nu_1(c(\mathcal{O}_F, T \cdot \mathcal{E}(1, \chi))).\end{aligned}$$

Then we have

$$(3.7) \quad \phi_1(T_{\mathfrak{l}}) = \nu_1(c(\mathcal{O}_F, T_{\mathfrak{l}} \cdot \mathcal{E}(1, \chi))) = \nu_1(c(\mathfrak{l}, \mathcal{E}(1, \chi))) = c(\mathfrak{l}, E_1(1, \chi_S)) = 1 + \chi_S(\mathfrak{l}).$$

Actually, we may regard ϕ_1 as

$$\phi_1: \mathbf{T} \otimes_{\Lambda} \Lambda_{(1)} \rightarrow E$$

since there exists $\mathcal{F} \in \mathcal{S}^o(\mathfrak{n}, \chi) \otimes_{\Lambda} \Lambda_{(1)}$ satisfying $\nu_1(\mathcal{F}) = t \cdot E_1(1, \chi_S)$.

- Let $\mathbf{T}_{(1)}$ be the localization of $\mathbf{T} \otimes_{\Lambda} \Lambda_{(1)}$ at $\text{Ker}(\phi_1)$, and $\mathcal{F}_{\mathbf{T}_{(1)}}$ the total ring of fractions of $\mathbf{T}_{(1)}$. As well known, there exists a basis of $\mathcal{S}^o(\mathfrak{n}, \chi) \otimes_{\Lambda} \Lambda'$ consisting of Hecke eigenforms if Λ' is large enough. Let $\mathcal{F}_1, \dots, \mathcal{F}_r$ be the elements of such

a basis which satisfy $\nu_1(\mathcal{F}_i) = E_1(1, \chi_S)$. We denote the Hecke eigenvalue of \mathcal{F}_i at $T \in \mathbf{T}_{(1)}$ by $\lambda_T(\mathcal{F}_i)$, the Hecke field of \mathcal{F}_i by $\mathbf{F}(\mathcal{F}_i)$. Then we can embed $\mathbf{T}_{(1)} \hookrightarrow \prod_{i=1}^r \mathbf{F}(\mathcal{F}_i)$ by $T \mapsto (\lambda_T(\mathcal{F}_i))_{i=1, \dots, r}$. Eventually, we can decompose $\mathcal{F}_{\mathbf{T}_{(1)}}$ into a product of fields $\mathcal{F}_{\mathbf{T}_{(1)}} = \mathbf{F}_1 \times \mathbf{F}_2 \times \dots \times \mathbf{F}_t$ with \mathbf{F}_i a finite extension field of \mathcal{F}_Λ . Take a factor $\mathbf{F} := \mathbf{F}_i$ of $\mathcal{F}_{\mathbf{T}_{(1)}}$. We denote by $T_{\mathfrak{l}}, U_{\mathfrak{l}}$ their images under the natural map $\mathbf{T} \rightarrow \mathcal{F}_{\mathbf{T}_{(1)}} \rightarrow \mathbf{F}$. The image of $\mathbf{T}_{(1)}$ under this map is denoted by \mathbf{R} . Then \mathbf{R} is a local ring with E the residue field. Let \mathfrak{m} be the maximal ideal of \mathbf{R} . Note that $\mathfrak{m} = \text{Ker } \phi_1: \mathbf{R} \rightarrow E$.

- We define the Λ -adic cyclotomic character $\underline{\epsilon}_{\text{cyc}}: G_F \rightarrow \Lambda^\times$ by

$$\nu_k(\underline{\epsilon}_{\text{cyc}}(\text{Frob}_{\mathfrak{l}})) = \langle N\mathfrak{l} \rangle^{k-1} \quad (\forall \mathfrak{l} \notin S_p).$$

Note that the p -adic cyclotomic character $\epsilon_{\text{cyc}}: G_F \rightarrow \mathbb{Z}_p^\times$ is characterized by $\epsilon_{\text{cyc}}(\text{Frob}_{\mathfrak{l}}) = N\mathfrak{l} \pmod{p}$ ($\forall \mathfrak{l} \notin S_p$).

We now introduce Wiles' "big Galois representation" [DDP, Theorem 4.1].

Theorem 3.14. *For each $\mathbf{F}(= \mathbf{F}_i)$, there exists a continuous irreducible Galois representation*

$$\rho(= \rho_i): G_F \rightarrow \text{GL}_2(\mathbf{F})$$

satisfying

1. *If $\mathfrak{l} \notin S$, then ρ is unramified at \mathfrak{l} and the characteristic polynomial of $\rho(\text{Frob}_{\mathfrak{l}})$ is*

$$X^2 - T_{\mathfrak{l}}X + \chi_{\underline{\epsilon}_{\text{cyc}}}(\text{Frob}_{\mathfrak{l}}).$$

2. *ρ is odd.*

3. *If $\mathfrak{q} \in S_p$, then*

$$\rho|_{G_{F_{\mathfrak{q}}}} \cong \begin{pmatrix} \chi_{\underline{\epsilon}_{\text{cyc}}} \eta_{\mathfrak{q}}^{-1} & * \\ 0 & \eta_{\mathfrak{q}} \end{pmatrix}.$$

Here we denote by $\eta_{\mathfrak{q}}$ the unramified character of $G_{F_{\mathfrak{q}}}$ characterized by

$$(3.8) \quad \eta_{\mathfrak{q}}(\text{Frob}_{\mathfrak{q}}) = U_{\mathfrak{q}}.$$

Now we prepare some properties of the representation ρ .

Theorem 3.15 ([DDP, Theorem 4.2]). *Let ρ be as in Theorem 3.14 and fix a complex conjugation $\delta \in G_F$. Since ρ is odd, we may assume that*

$$\rho(\delta) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

by replacing the \mathbf{F} -basis. We define continuous functions $a, b, c, d: G_F \rightarrow \mathbf{F}$ by

$$\rho(\sigma) = \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix}.$$

Take a change of basis matrix of Theorem 3.14-3. That is, for $\mathfrak{q} \in S_p$, there exists $\begin{pmatrix} A_{\mathfrak{q}} & B_{\mathfrak{q}} \\ C_{\mathfrak{q}} & D_{\mathfrak{q}} \end{pmatrix} \in \mathrm{GL}_2(\mathbf{F})$ satisfying

$$(3.9) \quad \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix} \begin{pmatrix} A_{\mathfrak{q}} & B_{\mathfrak{q}} \\ C_{\mathfrak{q}} & D_{\mathfrak{q}} \end{pmatrix} = \begin{pmatrix} A_{\mathfrak{q}} & B_{\mathfrak{q}} \\ C_{\mathfrak{q}} & D_{\mathfrak{q}} \end{pmatrix} \begin{pmatrix} \chi_{\mathfrak{E}_{\mathrm{cyc}}} \eta_{\mathfrak{q}}^{-1}(\sigma) & * \\ 0 & \eta_{\mathfrak{q}}(\sigma) \end{pmatrix} \quad (\forall \sigma \in G_{F_{\mathfrak{q}}}).$$

Then we have the following properties.

1. For all $\sigma \in G_F$, we have $a(\sigma), d(\sigma) \in \mathbf{R}^{\times}$, $a(\sigma) \equiv 1$, $d(\sigma) \equiv \chi(\sigma) \pmod{\mathbf{m}}$. That is,

$$\begin{aligned} \phi_1 \circ a &= 1, \\ \phi_1 \circ d &= \chi. \end{aligned}$$

2. $C_{\mathfrak{q}} \neq 0$ ($\forall \mathfrak{q} \in S_p - \{\mathfrak{p}\}$).

3. $b|_{G_{F_{\mathfrak{p}}}} \neq 0$. In particular, $A_{\mathfrak{p}} \neq 0$.

Proof. By assumption on δ , we see that

$$(3.10) \quad \begin{aligned} a(\sigma) &= \frac{1}{2}(\mathrm{Tr} \rho(\sigma) + \mathrm{Tr} \rho(\sigma\delta)), \\ d(\sigma) &= \frac{1}{2}(\mathrm{Tr} \rho(\sigma) - \mathrm{Tr} \rho(\sigma\delta)). \end{aligned}$$

On the other hand, by Theorem 3.14-1, (3.7) and the Chebotarev density theorem we have

$$\phi_1(\mathrm{Tr} \rho(\sigma)) = (1 + \chi)(\sigma).$$

Noting that $\chi(\delta) = -1$, we can write

$$\begin{aligned} \phi_1 \circ a(\sigma) &= \frac{1}{2}(\phi_1(\mathrm{Tr} \rho(\sigma)) + \phi_1(\mathrm{Tr} \rho(\sigma\delta))) = \frac{1}{2}((1 + \chi)(\sigma) + (1 + \chi)(\sigma\delta)) = 1, \\ \phi_1 \circ d(\sigma) &= \frac{1}{2}(\phi_1(\mathrm{Tr} \rho(\sigma)) - \phi_1(\mathrm{Tr} \rho(\sigma\delta))) = \frac{1}{2}((1 + \chi)(\sigma) - (1 + \chi)(\sigma\delta)) = \chi(\sigma) \end{aligned}$$

as desired. Next we see the upper left-hand entries of (3.9). Then we have

$$(3.11) \quad C_{\mathfrak{q}} b(\sigma) = A_{\mathfrak{q}} (\chi_{\mathfrak{E}_{\mathrm{cyc}}} \eta_{\mathfrak{q}}^{-1}(\sigma) - a(\sigma)).$$

Therefore $C_{\mathfrak{q}} = 0$ implies $A_{\mathfrak{q}} = 0$. Then the second statement follows from $\begin{pmatrix} A_{\mathfrak{q}} & B_{\mathfrak{q}} \\ C_{\mathfrak{q}} & D_{\mathfrak{q}} \end{pmatrix} \in \mathrm{GL}_2(\mathbf{F})$. Now we give a sketch of the proof of the third statement. Put

$$\mathbf{B} := \sum_{\sigma \in G_{F_{\mathfrak{p}}}} b(\sigma) \cdot \mathbf{R} \subset \mathbf{F}.$$

Actually, we see that \mathbf{B} is a finitely generated \mathbf{R} -module. Additionally, we put

$$\begin{aligned} K &: G_F \rightarrow \mathbf{B}, \\ K(\sigma) &:= \frac{b(\sigma)}{d(\sigma)}. \end{aligned}$$

Since we have $d(\sigma) \in \mathbf{R}^\times$ by the first statement, $\mathrm{Im}(K)$ generates \mathbf{B} over \mathbf{R} . Put $\overline{\mathbf{B}} := \mathbf{B}/\mathbf{mB}$. Then it is easy to check that the associated map

$$\begin{aligned} \overline{K} &: G_F \rightarrow \overline{\mathbf{B}}, \\ \overline{K}(\sigma) &:= K(\sigma) \bmod \mathbf{mB} \end{aligned}$$

is a cocycle $\in Z^1(G_F, \overline{\mathbf{B}}(\chi^{-1}))$ by noting that $a \equiv 1$, $d \equiv \chi \bmod \mathbf{m}$. Note that $\overline{\mathbf{B}}$ is an $E = \mathbf{R}/\mathbf{m}$ vector space generated by $\mathrm{Im}(\overline{K})$ and that $\overline{\mathbf{B}}(\chi^{-1}) := \overline{\mathbf{B}} \otimes_E E(\chi^{-1})$. Now we prove $b|_{G_{F_{\mathfrak{p}}}} \neq 0$ by contradiction. We see that $b = 0$ on $G_{F_{\mathfrak{p}}}$ implies $b = 0$ on G_F by the following technique, which we will use again later.

1. $\overline{K}|_{G_{F_v}}$ satisfies the following “local triviality properties” (a),(b). In particular, by (3.5), we have

$$[\overline{K}] \in H_{\mathfrak{p}, \mathrm{cyc}}^1(G_F, \overline{\mathbf{B}}(\chi^{-1})) = H_{\mathfrak{p}, \mathrm{cyc}}^1(G_F, E(\chi^{-1})) \otimes_E \overline{\mathbf{B}}.$$

- (a) For all $\mathfrak{q} \in S_p - \{\mathfrak{p}\}$, we have $[\overline{K}|_{G_{F_{\mathfrak{q}}}}] = 0 \in H^1(G_{F_{\mathfrak{q}}}, \overline{\mathbf{B}}(\chi^{-1}))$

Proof. By Theorem 3.15-2 and (3.11), we can write

$$K|_{G_{F_{\mathfrak{q}}}} = \frac{A_{\mathfrak{q}}}{C_{\mathfrak{q}}} \cdot \frac{\chi \epsilon_{\mathrm{cyc}} \eta_{\mathfrak{q}}^{-1} - a}{d}.$$

By definition of $\epsilon_{\mathrm{cyc}}, \eta_{\mathfrak{q}}$ and Theorem 3.15-1, we have

$$\begin{aligned} (3.12) \quad & \phi_1 \circ a = 1, \\ & \phi_1 \circ d = \chi, \\ & \phi_1 \circ \epsilon_{\mathrm{cyc}} = 1, \\ & \phi_1 \circ \eta_{\mathfrak{q}} = 1. \end{aligned}$$

Hence the assertion is clear. □

$$(b) \quad \overline{K}|_{G_{F_{\mathfrak{p}}}} = 0.$$

Proof. By our assumption $b|_{G_{F_{\mathfrak{p}}}} = 0$. \square

2. The fact that the class $[\overline{K}]$ is locally trivial at \mathfrak{p} (by 1-(b)) implies that the class $[\overline{K}]$ is globally trivial:

$$[\overline{K}] = 0 \in H^1(G_F, \overline{\mathbf{B}}(\chi^{-1})).$$

Proof. It follows from the injectivity of $\text{res}_{\mathfrak{p}}: H_{\mathfrak{p}, \text{cyc}}^1(G_F, \overline{\mathbf{B}}(\chi^{-1})) \hookrightarrow \overline{\mathbf{B}} \cdot \kappa_{\text{nr}} \oplus \overline{\mathbf{B}} \cdot \kappa_{\text{cyc}}$. \square

3. The fact that $[\overline{K}] = 0$ as a class implies that

$$\overline{K} = 0 \text{ as a function: } G_F \rightarrow \overline{\mathbf{B}}.$$

Proof. By $[\overline{K}] = 0$, we can write $\overline{K} = \theta \cdot (1 - \chi^{-1})$ with $\theta \in \overline{\mathbf{B}}$. Then we get

$$\theta = \theta \cdot (1 - \chi^{-1}(\delta))/2 = K(\delta)/2 = b(\delta)/2d(\delta) = 0.$$

Hence we get $\overline{K} = 0$. \square

4. Since $\overline{\mathbf{B}}$ is generated by elements $\in \text{Im}(\overline{K})$, $\overline{K} = 0$ implies $\overline{\mathbf{B}} = 0$, i.e., $\mathbf{B} = \mathbf{mB}$. Then Nakayama's Lemma states that $\mathbf{B} = 0$.

Therefore $b = 0$ if $b|_{G_{F_{\mathfrak{p}}}} = 0$. But it contradicts the irreducibility of ρ . The fact that $A_{\mathfrak{p}} \neq 0$ follows from (3.11), $b|_{G_{F_{\mathfrak{p}}}} \neq 0$ and $\begin{pmatrix} A_{\mathfrak{p}} & B_{\mathfrak{p}} \\ C_{\mathfrak{p}} & D_{\mathfrak{p}} \end{pmatrix} \in \text{GL}_2(\mathbf{F})$. \square

§ 3.5. The weight “ $1 + \varepsilon$ ” specialization.

The weight 1 specialization $\nu_1: \Lambda_{(1)} \rightarrow E$ induces the weight $1 + \varepsilon$ specialization

$$\nu_{1+\varepsilon}: \Lambda_{(1)} \rightarrow \Lambda_{(1)}/(\text{Ker } \nu_1)^2 \cong E[\varepsilon]/(\varepsilon^2).$$

More explicitly, considering any element $f \in \Lambda_{(1)}$ as a meromorphic function $f(s)$ on \mathbb{Z}_p , we define

$$\nu_{1+\varepsilon}(f) := f(1) + f'(1)\varepsilon \in E[\varepsilon]/(\varepsilon^2) = E \oplus E\varepsilon.$$

Definition 3.16. We define two characters

$$\psi_1, \psi_2: G_F \rightarrow E[\varepsilon]/(\varepsilon^2)^\times$$

by

$$\begin{aligned} \psi_1 &:= 1 + v(1)\kappa_{\text{cyc}} \cdot \varepsilon, \\ \psi_2 &:= \chi \cdot (1 + u(1)\kappa_{\text{cyc}} \cdot \varepsilon). \end{aligned}$$

Note that these characters are lifts of $1, \chi: G_F \rightarrow E^\times$, respectively. Then we have

- ψ_1 is unramified outside of S_p and satisfies

$$\psi_1(\text{Frob}_{\mathfrak{l}}) = 1 + v(1)\kappa_{\text{cyc}}(\text{Frob}_{\mathfrak{l}})\varepsilon \quad (\forall \mathfrak{l} \notin S_p).$$

- ψ_2 is unramified outside of S and satisfies

$$\psi_2(\text{Frob}_{\mathfrak{l}}) = \chi(\mathfrak{l}) (1 + u(1)\kappa_{\text{cyc}}(\text{Frob}_{\mathfrak{l}})\varepsilon) \quad (\forall \mathfrak{l} \notin S).$$

As usual, we consider ψ_1, ψ_2 as multiplicative functions on the set of ideals by the rule of

$$\begin{aligned} \psi_1(\mathfrak{q}) &= 1 \quad (\forall \mathfrak{q} \in S_p), \\ \psi_2(\mathfrak{l}) &= 0 \quad (\forall \mathfrak{l} \in S). \end{aligned}$$

We defined the ordinary Λ -adic Hilbert modular form \mathcal{H} by modifying a product of Eisenstein series. Therefore \mathcal{H} is not necessarily an eigenform. Nevertheless, instead of \mathcal{H} , we can show that $\mathcal{H} \bmod (\text{Ker } \nu_1)^2$ is an eigenform. It was shown by the explicit calculation of Fourier coefficients. Namely, the following results are obtained in [DDP, Proposition 3.6].

Proposition 3.17. *Let \mathcal{H} be as in Proposition 3.13. Consider the weight $1 + \varepsilon$ specialization $\mathcal{H}_{1+\varepsilon} := \nu_{1+\varepsilon}(\mathcal{H})$. ($\mathcal{H}_{1+\varepsilon}$ is the family of formal Fourier coefficients $c(\mathfrak{m}, \mathcal{H}_{1+\varepsilon}) := \nu_{1+\varepsilon}(c(\mathfrak{m}, \mathcal{H}))$, $c(\lambda, \mathcal{H}_{1+\varepsilon}) := \nu_{1+\varepsilon}(c(\lambda, \mathcal{H}))$.) The action of a Hecke operator $T \in \tilde{\mathbf{T}}$ is defined by*

$$c(\mathfrak{m}, T\mathcal{H}_{1+\varepsilon}) := \nu_{1+\varepsilon}(c(\mathfrak{m}, T\mathcal{H})).$$

Then we have the following.

- $\mathcal{H}_{1+\varepsilon}$ is a simultaneous eigenform for the Hecke operators. Note that its eigenvalues belong to $E[\varepsilon]/(\varepsilon^2)$.
- $c(1, \mathcal{H}_{1+\varepsilon}) = 1$.
- $c(\mathfrak{l}, \mathcal{H}_{1+\varepsilon}) = \psi_1(\mathfrak{l}) + \psi_2(\mathfrak{l}) \quad (\forall \mathfrak{l} \neq \mathfrak{p})$.
- $c(\mathfrak{p}, \mathcal{H}_{1+\varepsilon}) = 1 + w'(1)\varepsilon$.

Therefore we have the $\Lambda_{(1)}$ -algebra homomorphism

$$\begin{aligned} \phi_{1+\varepsilon}: \tilde{\mathbf{T}} \otimes_{\Lambda} \Lambda_{(1)} &\rightarrow E[\varepsilon]/(\varepsilon^2), \\ T_{\mathfrak{l}} &\mapsto c(\mathfrak{l}, \mathcal{H}_{1+\varepsilon}) \quad (\mathfrak{l} \notin S), \\ U_{\mathfrak{l}} &\mapsto c(\mathfrak{l}, \mathcal{H}_{1+\varepsilon}) \quad (\mathfrak{l} \in S). \end{aligned}$$

Actually, $\phi_{1+\varepsilon}$ factors through the quotient

$$\phi_{1+\varepsilon}: \mathbf{T} \otimes_{\Lambda} \Lambda_{(1)} \rightarrow E[\varepsilon]/(\varepsilon^2),$$

since we can write $\mathcal{F} = T\mathcal{H} \in \mathcal{S}^o(\mathbf{n}, \chi) \otimes_{\Lambda} \Lambda_{(1)}$ with an element $T \in \tilde{\mathbf{T}}$. Note that $\phi_{1+\varepsilon}$ is a lift of

$$\phi_1: \mathbf{T} \otimes_{\Lambda} \Lambda_{(1)} \rightarrow E.$$

Proof. For simplicity, we give the proof for the case of $|S_p| > 1$. (In the case of $|S_p| = 1$, the proof is similar but more complicated. See [DDP, Proof of Proposition 3.6]). By using that $\mathcal{H} = u\mathcal{E}(1, \chi) + w\mathcal{P}^o$, $u(1) = 1$, $w(1) = 0$, $u'(1) + w'(1) = 0$, $\nu_1(\mathcal{E}(1, \chi)) = E_1(1, \chi_S)$, $\nu_1(\mathcal{P}^o) = E_1(1, \chi_R)$, we get

$$\mathcal{H}_{1+\varepsilon} = \nu_{1+\varepsilon}(\mathcal{E}(1, \chi)) + w'(1)(E_1(1, \chi_R) - E_1(1, \chi_S)).$$

We will write down the \mathbf{m} -th Fourier coefficient of this. We define an integral ideal \mathbf{m}_0 and non-negative integers $\text{ord}_{\mathfrak{q}} \mathbf{m}$ by $\mathbf{m} = \mathbf{m}_0 \prod_{\mathfrak{q} \in S_p} \mathfrak{q}^{\text{ord}_{\mathfrak{q}} \mathbf{m}}$, $\gcd(\mathbf{m}_0, (p)) = 1$. Then we have

$$c(\mathbf{m}, \nu_{1+\varepsilon}(\mathcal{E}(1, \chi))) = \nu_{1+\varepsilon}(c(\mathbf{m}, \mathcal{E}(1, \chi))) = \sum_{\mathfrak{r}|\mathbf{m}_0} \chi(\mathfrak{r})(1 + \kappa_{\text{cyc}}(\mathfrak{r})\varepsilon)$$

since $\nu_k(c(\mathbf{m}, \mathcal{E}(1, \chi))) = \sum_{\mathfrak{r}|\mathbf{m}_0} \chi(\mathfrak{r})\langle N\mathfrak{r} \rangle^{k-1}$. By noting that $\chi_R(\mathfrak{r}) - \chi_S(\mathfrak{r}) = 0$ if $\mathfrak{p} \nmid \mathfrak{r}$ and that $\chi_R(\mathfrak{r}) - \chi_S(\mathfrak{r}) = \chi_R(\mathfrak{r}) = \chi_R(\mathfrak{r}/\mathfrak{p})$ if $\mathfrak{p}|\mathfrak{r}$, we can write

$$c(\mathbf{m}, E_1(1, \chi_R)) - c(\mathbf{m}, E_1(1, \chi_S)) = \sum_{\mathfrak{r}|\mathbf{m}} (\chi_R(\mathfrak{r}) - \chi_S(\mathfrak{r})) = \text{ord}_{\mathfrak{p}} \mathbf{m} \sum_{\mathfrak{r}|\mathbf{m}_0} \chi(\mathfrak{r}).$$

Consequently, we get

$$(3.13) \quad c(\mathbf{m}, \mathcal{H}_{1+\varepsilon}) = \left(\sum_{\mathfrak{r}|\mathbf{m}_0} \psi_1(\mathbf{m}_0/\mathfrak{r})\psi_2(\mathfrak{r}) \right) \times (1 + \omega'(1)\varepsilon)^{\text{ord}_{\mathfrak{p}} \mathbf{m}},$$

where $\psi_1 = 1$ in this case. Therefore the fact that $\mathcal{H}_{1+\varepsilon}$ is a simultaneous eigenform can be shown similarly to the case of the usual Eisenstein series $E_k(\eta, \psi)$ whose \mathbf{m} -th Fourier coefficient is $c(\mathbf{m}, E_k(\eta, \psi)) = \sum_{\mathfrak{r}|\mathbf{m}} \eta(\mathbf{m}/\mathfrak{r})\psi(\mathfrak{r})N\mathfrak{r}^{k-1}$. The remaining assertions also follow from (3.13). \square

We note that we have $\omega'(1) = u(1)\mathcal{L}_{\text{an}}(\chi)$ since $\omega(k) = u(k)\mathcal{L}_{\text{an}}(k, \chi)$. Then we get the following main result in this subsection.

Theorem 3.18 ([DDP, Theorem 3.7]). *Under the assumption (3.1):*

If $|S_p| > 1$, assume that Leopoldt's conjecture is true for F .

If $|S_p| = 1$, assume that Leopoldt's conjecture is true for F ,

and that $\text{ord}_{s=1}(\mathcal{L}_{\text{an}}(s, \chi) + \mathcal{L}_{\text{an}}(s, \chi^{-1})) = \text{ord}_{s=1}\mathcal{L}_{\text{an}}(s, \chi^{-1})$,

there exists a homomorphism

$$\phi_{1+\varepsilon}: \mathbf{T}_{(1)} \rightarrow E[\varepsilon]/(\varepsilon^2)$$

satisfying

$$\begin{aligned} \phi_{1+\varepsilon}(T_{\mathfrak{l}}) &= \psi_1(\mathfrak{l}) + \psi_2(\mathfrak{l}) \quad (\forall \mathfrak{l} \notin S), \\ \phi_{1+\varepsilon}(U_{\mathfrak{l}}) &= \psi_1(\mathfrak{l}) + \psi_2(\mathfrak{l}) \quad (\forall \mathfrak{l} \in S), \\ &= \begin{cases} \psi_1(\mathfrak{l}) & (\forall \mathfrak{l} \in R), \\ 1 + u(1)\mathcal{L}_{\text{an}}(\chi)\varepsilon & (\mathfrak{l} = \mathfrak{p}). \end{cases} \end{aligned}$$

§ 3.6. Construction of a cocycle.

Consider the product of Galois representations ρ_i in Theorem 3.14

$$\rho_{(1)} := \prod_i \rho_i: G_F \rightarrow \text{GL}_2(\mathcal{F}_{\mathbf{T}_{(1)}}).$$

Taking the basis as in Theorem 3.15, we write

$$\begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix} := \rho_{(1)}(\sigma).$$

We summarize the properties of continuous maps $a, b, c, d: G_F \rightarrow \mathcal{F}_{\mathbf{T}_{(1)}}$ which we have seen:

- $a(\sigma), d(\sigma) \in \mathbf{T}_{(1)}^\times$ ($\forall \sigma \in G_F$) (by Theorem 3.15-1).
- By Theorem 3.14-3, for each $\mathfrak{q} \in S_p$, there exists $\begin{pmatrix} A_{\mathfrak{q}} & B_{\mathfrak{q}} \\ C_{\mathfrak{q}} & D_{\mathfrak{q}} \end{pmatrix} \in \text{GL}_2(\mathcal{F}_{\mathbf{T}_{(1)}})$ satisfying

$$\begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix} \begin{pmatrix} A_{\mathfrak{q}} & B_{\mathfrak{q}} \\ C_{\mathfrak{q}} & D_{\mathfrak{q}} \end{pmatrix} = \begin{pmatrix} A_{\mathfrak{q}} & B_{\mathfrak{q}} \\ C_{\mathfrak{q}} & D_{\mathfrak{q}} \end{pmatrix} \begin{pmatrix} \chi_{\in \text{cyc}} \eta_{\mathfrak{q}}^{-1}(\sigma) & * \\ 0 & \eta_{\mathfrak{q}}(\sigma) \end{pmatrix} \quad (\forall \sigma \in G_{F_{\mathfrak{q}}}).$$

Moreover we see that $A_{\mathfrak{p}} \in \mathcal{F}_{\mathbf{T}_{(1)}}^\times$ (by Theorem 3.15-3), $C_{\mathfrak{q}} \in \mathcal{F}_{\mathbf{T}_{(1)}}^\times$ ($\forall \mathfrak{q} \in S_p - \{\mathfrak{p}\}$) (by Theorem 3.15-2).

Therefore by putting

$$\begin{aligned} K(\sigma) &:= \frac{C_{\mathfrak{p}}}{A_{\mathfrak{p}}} \frac{b(\sigma)}{d(\sigma)} \quad (\sigma \in G_F), \\ x_{\mathfrak{q}} &:= \frac{C_{\mathfrak{p}}}{A_{\mathfrak{p}}} \frac{A_{\mathfrak{q}}}{C_{\mathfrak{q}}} \quad (\mathfrak{q} \in S_p - \{\mathfrak{p}\}), \\ \mathbf{B} &:= \sum_{\sigma \in G_F} \frac{C_{\mathfrak{p}}}{A_{\mathfrak{p}}} b(\sigma) \cdot \mathbf{T}_{(1)} \subset \mathcal{F}_{\mathbf{T}_{(1)}}, \end{aligned}$$

we can write

$$(3.14) \quad K|_{G_{F_q}} = \begin{cases} x_q \cdot \frac{\chi_{\text{cyc}} \eta_q^{-1} - a}{d} & (q \in S_p - \{p\}) \\ \frac{\epsilon_{\text{cyc}} \eta_q^{-1} - a}{d} & (q = p). \end{cases}$$

As preparation, we prove a claim in the proof of [DDP, Theorem 4.4]. Put $\mathbf{m} = \text{Ker } \phi_1: \mathbf{T}_{(1)} \rightarrow E$.

Lemma 3.19. *We have $\mathbf{B} \subset \mathbf{m}$.*

Proof. Put $\mathbf{B}^\sharp := (\mathbf{B} + \mathbf{m})/\mathbf{m}$, $\overline{\mathbf{B}}^\sharp := \mathbf{B}^\sharp/\mathbf{m}\mathbf{B}^\sharp$, and $\overline{K}^\sharp: G_F \rightarrow \overline{\mathbf{B}}^\sharp$ to be the associated map to $K: G_F \rightarrow \mathbf{B}$. Then the local triviality of \overline{K}^\sharp implies the global triviality of \overline{K}^\sharp . This can be seen similarly as in the proof of Theorem 3.15-3. In fact, we see the following.

1. $[\overline{K}^\sharp|_{G_{F_q}}] = 0$ ($\forall q \in S_p - \{p\}$), $\overline{K}^\sharp|_{G_{F_p}} = 0$. In particular, by (3.5), we see that $[\overline{K}^\sharp] \in H_{p, \text{cyc}}^1(G_F, \overline{\mathbf{B}}^\sharp(\chi^{-1})) = H_{p, \text{cyc}}^1(G_F, E(\chi^{-1})) \otimes_E \overline{\mathbf{B}}^\sharp$.

Proof. It follows from (3.12), (3.14). □

2. $[\overline{K}^\sharp|_{G_{F_p}}] = 0$ implies $[\overline{K}^\sharp] = 0$.

Proof. It is clear since res_p is injective on $H_{p, \text{cyc}}^1(G_F, \overline{\mathbf{B}}^\sharp(\chi^{-1}))$. □

3. $[\overline{K}^\sharp] = 0$ implies $\overline{K}^\sharp = 0$.

Proof. Similarly to the proof of step 3 in the proof of Theorem 3.15-3. □

4. $\overline{K}^\sharp = 0$ implies $\overline{\mathbf{B}}^\sharp = \mathbf{B}^\sharp/\mathbf{m}\mathbf{B}^\sharp = 0$. That is, $\mathbf{B}^\sharp = \mathbf{m}\mathbf{B}^\sharp$.

Now Nakayama's Lemma states $\mathbf{B}^\sharp = 0$, so we get $\mathbf{B} + \mathbf{m} = \mathbf{m}$. Hence the assertion is clear. □

The fact that $\mathbf{B} \subset \mathbf{m}$ implies $\phi_1 \circ K = 0$, so $(\phi_{1+\varepsilon} \circ K)(G_F) \subset E \cdot \varepsilon$. That is, there exists a continuous map $\kappa: G_F \rightarrow E$ satisfying

$$(3.15) \quad \kappa \cdot \varepsilon = \phi_{1+\varepsilon} \circ K.$$

We have

$$\begin{aligned} x_q &\in \mathbf{B} \subset \mathbf{m}, \\ \phi_{1+\varepsilon} \circ \epsilon_{\text{cyc}} &= 1 + \kappa_{\text{cyc}} \varepsilon \end{aligned}$$

by definition and the above Lemma. Moreover we see that

$$\phi_{1+\varepsilon} \circ \eta_{\mathfrak{p}} = 1 + u(1)\mathcal{L}_{\text{an}}(\chi)\kappa_{\text{nr}}\varepsilon$$

by (3.8), Theorem 3.18, and that

$$\begin{aligned}\phi_{1+\varepsilon} \circ a &= \psi_1 = 1 + v(1)\kappa_{\text{cyc}}\varepsilon, \\ \phi_{1+\varepsilon} \circ d &= \psi_2 = \chi(1 + u(1)\kappa_{\text{cyc}}\varepsilon)\end{aligned}$$

by (3.10), Theorem 3.14-1, Theorem 3.18, Definition 3.16. Summarizing the above, we have the following Theorem.

Theorem 3.20 ([DDP, Theorem 4.4]). *We have*

$$\begin{aligned}[\kappa|_{G_{F_{\mathfrak{q}}}}] &= 0 \in H^1(G_{F_{\mathfrak{q}}}, E(\chi^{-1})) \quad (\forall \mathfrak{q} \in S_p - \{\mathfrak{p}\}), \\ \kappa|_{G_{F_{\mathfrak{p}}}} &= u(1)(-\mathcal{L}_{\text{an}}(\chi) \cdot \kappa_{\text{nr}} + \kappa_{\text{cyc}}).\end{aligned}$$

Proof. By (3.14) and $x_{\mathfrak{q}} \in \mathbf{m}$, we can write

$$\kappa|_{G_{F_{\mathfrak{q}}}} = x'_{\mathfrak{q}} \cdot \phi_1 \circ \frac{\chi \varepsilon_{\text{cyc}} \eta_{\mathfrak{q}}^{-1} - a}{d} = x'_{\mathfrak{q}} \cdot (1 - \chi^{-1})$$

with an element $x'_{\mathfrak{q}} \in E$ for $\mathfrak{q} \in S_p - \{\mathfrak{p}\}$. Then the first assertion is clear. Similarly we can write

$$\begin{aligned}\kappa|_{G_{F_{\mathfrak{p}}}} \cdot \varepsilon &= \phi_{1+\varepsilon} \circ \frac{\varepsilon_{\text{cyc}} \eta_{\mathfrak{q}}^{-1} - a}{d} \\ &= \frac{(1 + \kappa_{\text{cyc}} \cdot \varepsilon)(1 - u(1)\mathcal{L}_{\text{an}}(\chi)\kappa_{\text{nr}} \cdot \varepsilon) - (1 + v(1)\kappa_{\text{cyc}} \cdot \varepsilon)}{1 + u(1)\kappa_{\text{cyc}} \cdot \varepsilon} \\ &= (-u(1)\mathcal{L}_{\text{an}}(\chi)\kappa_{\text{nr}} + \kappa_{\text{cyc}} - v(1)\kappa_{\text{cyc}}) \cdot \varepsilon.\end{aligned}$$

By definition of u, v in Corollary 3.10 and the assumption (3.1), we have

$$\begin{aligned}u(1) &= \begin{cases} 1 & \text{if } |S_p| > 1, \\ \frac{\mathcal{L}_{\text{an}}^{(t)}(1, \chi^{-1})}{\mathcal{L}_{\text{an}}^{(t)}(1, \chi) + \mathcal{L}_{\text{an}}^{(t)}(1, \chi^{-1})} & \text{if } |S_p| = 1, \end{cases} \\ v(1) &= \begin{cases} 0 & \text{if } |S_p| > 1, \\ \frac{\mathcal{L}_{\text{an}}^{(t)}(1, \chi)}{\mathcal{L}_{\text{an}}^{(t)}(1, \chi) + \mathcal{L}_{\text{an}}^{(t)}(1, \chi^{-1})} & \text{if } |S_p| = 1 \end{cases}\end{aligned}$$

with $t := \text{ord}_{s=1} \mathcal{L}_{\text{an}}(s, \chi^{-1})$. Then the second assertion is clear. \square

By (3.5) and the above Theorem, we see that $\kappa \in H_{\mathfrak{p}, \text{cyc}}^1(G_F, E(\chi^{-1}))$ and that Conjecture 3.3 holds true under the assumption (3.1).

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